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Theory Of Machines.

Motors are such machines or devices which cause the various forces Nature to perform various kinds of work. These natural forces are:

(1). The muscular force exerted by animals;
(2). Water acting by its weight;
(3). The force exerted by moving air, or wind;
and (4). Heat.

(1.) Muscular force machines, as e.g. the treadmill;
(2). Hydraulic Motors.
(3.) Wind Motors, and
(4). Caloric Machines or motors.

All force might be reduced to the effect or action of heat, such as the heat of the sun, and hence all machines might be called Caloric machines. Thus, Muscular force is due to the combustion of nutriment in the body, which food in turn is ripened or rendered fit for consumption by the action of the sun. Water, again is produced by the condensation of the aqueous vapor generated by the sun's heat, and falling down upon the mountains produces streams which may be adapted to moving machines. Wind, also, has its origin in the same source, due to the rarefaction by heat of one mass, thereby causing the motion of a cooler mass to restore equilibrium. Caloric machines proper are that class of motors which are adapted to the utilization of heated gases or vapors. A steam engine is therefore only a particular kind of Caloric machine.

As muscular force can not be weighed or measured, or only so by its effects, it can not be reduced to a scientific basis, being in the highest sense empirical. It may therefore be neglected

and we will proceed directly to the consideration of the second class of motors, which ranks among the most important.

Hydraulic Motors. — The water which moves the machines must first be collected, and is generally taken from streams. To obtain the full benefit of the natural fall, it often becomes necessary to concentrate as much as possible this latter, which is accomplished by means of Dams. As it is usually impracticable to apply the motor at the dam directly, the water supply is generally conveyed to the machine thro' a canal, race, or pipes, and after it has performed its task, removed in a similar manner.

Above the motor, the supply of water is carried either in pipes or some kind of open channel or canal. It thus becomes necessary to consider first the laws attending the damming of streams and its subsequent flow thro artificial canals.

Channels (Wasserläufe) are either natural or artifical, and the when open, the latter are called canals, especially if they are constructed with considerable care and with suitable mechanical structures.

The theory of the flow of water is applicable to natural as well as artificial channels, only that in the former, the uniformity of its laws appear to be impaired by all kinds of irregularities of the channel itself as well as by other extraneous causes, while in the latter, these laws seem to act with great uniformity due to the regular form of the channel.

To arrive at these laws, we assume that the body of flowing water may be divided by vertical planes into separate bodies against whose end surfaces or cross-sections produced by these planes, the velocity of the water is normally directed in all points. In a rectilinear canal, all these cross-sections are of course parallel.

The channel is cut by one of these vertical planes perpendicular to its axis into a figure called the "cross-section or profile of the channel", as ABBC, Fig 1.

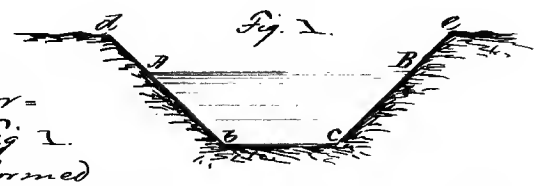


Fig. 1.

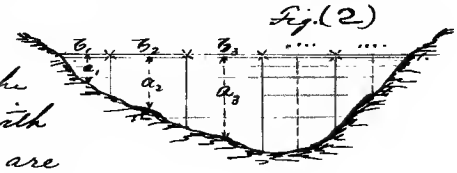
The three sides of the trapezoid formed by the section of the body of water which are in contact with the channel are together called the "wetted cross-section or profile" (benetzetem quer-profil), while the fourth line AB is named the "cross-section of the water" itself (quer-profil des Wassers). The wetted cross-section and the cross-section of the water together constitute the "cross-section of the body of water" (Quer-profil des Wasser-Körpers).

The difference in level between any two cross-sections of the water is called the fall or head (gefälle); and if (H) is the fall between any two cross-sections at any distance apart, and (h) the fall between any two whose distance apart is unity $=1$, the ratio $\frac{H}{h}$ is called the relative fall (relativ Gefälle); or it is the ratio of $\left(\frac{dy}{dx}\right)$ where x & y denote the abscissa and ordinate

By the "declivity" (abhang) is understood that angle which the tangent of the longitudinal section at the given point makes with the horizon.

The quotient $\frac{F}{b}$, where F denotes the area of the entire cross-section, and (b) the breadth or "the cross section of the water", is called the "mean depth of the water" (Mittlere Wasser-Tiefe) at that section. It is generally found by dividing the "profile of the water" into a number of points and then measuring the depth at those points. We thus have with sufficient accuracy, if the points $b_1, b_2 \cdots$ are close enough together:

Fig. (2)

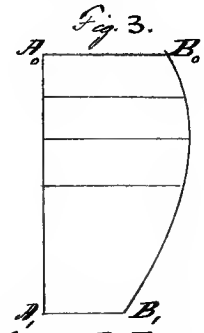


$$F = a_1 b_1 + a_2 b_2 + \cdots$$

The velocity of the water increases up to a certain point as we proceed from the bottom to the surface of the water, due to the friction upon its bed, and also to the resistance to displacement of two adjacent particles of water.

The geometrical position of the line of greatest velocity is called (strom-strich). The velocity furthermore varies more rapidly in a vertical than in a horizontal direction.

Fig. 3.



Let $A_0 A_1$ be any vertical line thro' the water, and
" $A_0 B_0$ = the velocity at the surface, and
" $A_1 B_1$ = " " " " bed.
Were the remaining ordinates interpolated between A_0 and A_1, and their extremities joined, we should obtain a curve $B_0 B_1$, which is called the "curve of the velocity" (Geschwendigkeits Curve).

Let (v) = the velocity at any point of a vertical line $A_0 A_1$;
" (v) = " mean velocity " " " " " ; and
" (C) = " " " " of the entire cross-section.
We then have, if Q denotes the quantity of water flowing through this cross-section in a unit of time:

$$C = \frac{Q}{F}.$$

Conceive the entire cross-section to be divided into band-like areas $F_1, F_2, \ldots$, with the mean velocities $c_1, c_2, c_3 \ldots$. We may then place:

$$Q = F_1 c_1 + F_2 c_2 + \ldots\ldots,$$

understanding by $c_1, c_2 \ldots$ the mean velocities of the vertical lines which divide the areas $F_1, F_2, \ldots$ each into two equal halves; hence we may also place:

$$Q = a_1 b_1 c_1 + a_2 b_2 c_2 + \ldots\ldots$$

To find the quantity of water Q, it is however necessary to know the law of variation of the velocities from B_0 to B_1.

Water is said to be in a state of steady flow (Beharrungs-zustand) throughout a given distance, when for every cross-section in this distance, Q, F and (c) are constant, or even if the product of F and (c) is constant, although the two factors themselves may vary. In the latter case, the water is said to be in a [illegible] state of steady flow (Beharrungszustand) which is the former, in a similar state of steady flow (Beharrungszustande)

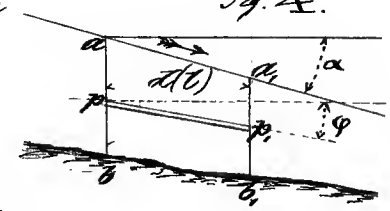
It now becomes necessary to understand the law of variation of the velocities in a vertical line thro' the water, and also the general relations between (c) and the dimensions of the cross-section, and the relative fall. The principle announced by Dubuat has this for its object. Dubuat's Principle is, that the moving force of the water depends only upon its relative fall, or the inclination of its surface to the horizon.

To fix this idea, conceive the whole area of the cross-section to be divided into small cross-sections of elementary threads of water, and let $(p_1 - p_1)$, Fig. 4., be such a thread.

Further, let: (α) be the inclination of the surface to the horizon, and:

(φ) = the inclination of the thread p, p_1, to the horizon.

Fig. 4.



The entire moving force with which this thread is moved along in its direction is the sum of all the moving forces acting upon it, viz: the component of the force of gravity in the direction $(p\text{-}p_1)$ and the difference of the pressures of the masses of water behind at (p) and in front at (p_1).

If the length of the thread is (dl) and its cross-section $= dF$, we then

have its Volume $= d\Sigma . dl$, and its weight $= \gamma . d\Sigma . dl$, where (γ) denotes the specific gravity of water, or its weight per unit of volume. Again, let (φ) denote the [illegible] inclination of the thread, or the weight which tends to move it in the direction $p_1 \to p$, and let

$\gamma . d\Sigma (ap)$ = the pressure behind the thread at p, and

$\gamma . d\Sigma (a_1 p_1)$ = the pressure in front of the thread at p_1.

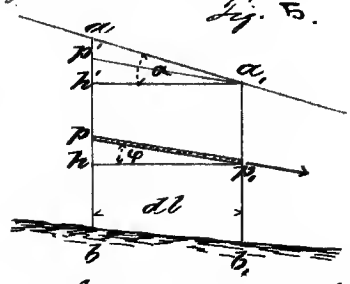


Fig. 5.

We then have from the figure:

$dl . \varphi = ph$, as the arc is sensibly = to the sine for small angles; and the moving force:

$$\gamma . d\Sigma . dl . \varphi + \gamma . d\Sigma (ap - a_1 p_1) = \gamma . d\Sigma (ph + ap') =$$
$$\gamma . d\Sigma . ah' = \gamma . d\Sigma . dl . \alpha ;$$

the entire moving force for the whole cross-section is therefore

$$= \gamma . dl . \alpha . \int d\Sigma = \gamma . \Sigma . dl . \alpha = mg . \alpha ,$$

where (m) denotes the mass of the volume of water and (g) the acceleration of the force of gravity.

Let us next take two cross-sections Σ, Σ_1, whose distance apart is unity $= 1$, which however, may be taken as small as desired, and regard the water between them to be in a state of uniform permanence (gleichförmiger Beharrungszustand). Take the origin of coordinates at any arbitrary point O, Fig. 6., and the axes OX and OY. Conceive an infinitely small rectangle ($dy . dx$) whose coordinates are (x) and (y). The moving forces acting upon the thread, whose cross-section is this small rectangle, are then the weight of the thread itself, the weight of those behind and in front of it, and the action of the 4 enclosing threads. Its volume is therefore:

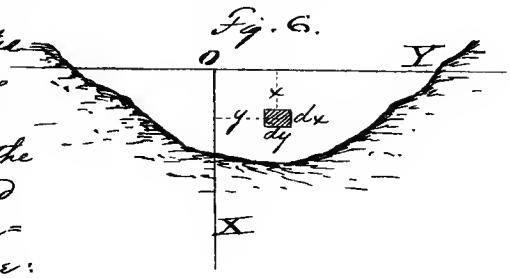


Fig. 6.

$$= dx . dy . 1 ;$$

its weight: $= \gamma . dx . dy . 1$, and

its moving force: $= \gamma . dx . dy . 1 . \alpha$.

If the velocity of the thread in question is $= (v)$, that of the one above it will then be $= v - \frac{dv}{dx} . dx$; hence $-\frac{dv}{dx} . dx$ is the relative velocity of the upper to the lower thread, as this increase or diminution of velocity in two adjacent elements is expressed analytically by the first differential coefficient taken with respect to the proper variable, and may be either positive or negative depending upon the direction in which the coordinate is taken.

Accordingly, as the velocities of the threads above and below the one considered vary, there will be generated a kind of shearing or sliding force, which may be expressed as a function of the surfaces of contact and the relative velocity.

But the surface of contact is $= dy.1$; hence the force with which the upper thread acts upon (pp_1) is:

$$f(-\tfrac{dv}{dx}).dy.1;$$

while the action of the thread immediately below (pp_1), or the acceleration of the latter upon the lower thread, is expressed by:

$$\left[f(-\tfrac{dv}{dx}) + \frac{df(-\frac{dv}{dx})}{dx}dx\right]dy.1.$$

The forces of the four threads surrounding (pp_1) are thus

$$= \frac{d}{dx}f(-\tfrac{dv}{dx}).dx.dy + \frac{d}{dy}.f(-\tfrac{dv}{dy}).dxdy,$$

which, in order that there may be a state of permanence, must be

$$= \gamma.\alpha.dxdy, \text{ or:}$$

$$\frac{d}{dx}f(-\tfrac{dv}{dx}) + \frac{d}{dy}.f(-\tfrac{dv}{dy}) = \gamma.\alpha$$

If the <u>breadth</u> of the body of water is much greater than its <u>depth</u>, the variations in velocity in the former direction will be extremely slight as compared with those in a vertical direction. We may therefore neglect the second term of the first member of the last equation in comparison to the first term, and hence place:

$$\frac{d}{dx}f(-\tfrac{dv}{dx}) = \gamma.\alpha.$$

But as we do not yet know what kind of a function $f(-\frac{dv}{dx})$ is, some supposition must be made which must afterwards be verified by experiment. Let us therefore place:

$$f(-\tfrac{dv}{dx}) = m(-\tfrac{dv}{dx})^n,$$

where (m) and (n) denote quantities to be determined experimentally provided that this is the true form of the function.
Dividing by (m), we have, from the previous equation:

$$\frac{d}{dx}.(-\tfrac{dv}{dx})^n = \gamma.\frac{\alpha}{m}, \text{ which, upon integration, gives:}$$

$$(-\tfrac{dv}{dx})^n = \gamma.\frac{\alpha}{m}.x + Const.$$

Let (a_0) be the coördinate of that point where the value of (v) is a maximum, i.e., for which $\frac{dv}{dx} = 0$. We then have:

$$0 = \frac{\alpha\gamma}{m}.a_0 + Const., \text{ and: } Constant = -\frac{\alpha\gamma}{m}.a_0;$$

hence:

$$(-\tfrac{dv}{dx})^n = \frac{\alpha\gamma}{m}(x - a_0), \text{ and}$$

$$\frac{dv}{dx} = -\left(\frac{\alpha\gamma}{m}\right)^{\frac{1}{n}}.(x-a_0)^{\frac{1}{n}}, \text{ which upon integration gives; — when the maximum velocity is denoted by } (v'):$$

$$v = v' - \left(\frac{\alpha\gamma}{m}\right)^{\frac{1}{n}}\frac{(x-a_0)^{\frac{1}{n}+1}}{\frac{1}{n}+1},$$

or: $$v = v' - \frac{n}{n+1}\left(\frac{\alpha \gamma}{m}\right)^{\frac{1}{n}} \cdot \left(x - a_0\right)^{\frac{n+1}{n}}.$$

These values of (m) and (n) must now be compared with the results found from experiments. In the course of their extensive experiments on the Mississippi River, Messrs. Humphreys and Abbott found that the curve of velocity coincides very nearly with a parabola having its axis horizontal and in the position where the greatest velocity occurs ($A'B'$, Fig. 7).

For the purpose of comparison, we will accordingly place: v = the velocity at any point below the surface $A_0 B_0$. Taking the axis $A'B''$ and the tangent at the point B' for the co-ordinate axes, the equation of a parabola would be:

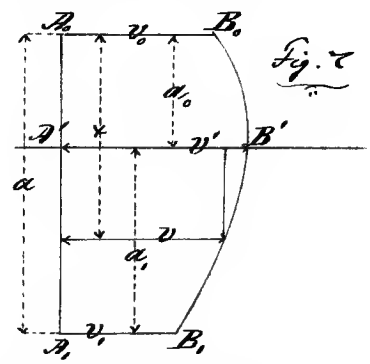


Fig. 7

$$(x - a_0)^2 = C(v' - v),$$

or

$$v = v' - \frac{(x - a_0)^2}{C} \quad \ldots\ldots \quad (1.)$$

If the equation found above were true, we should now have $\frac{n+1}{n} = 2$, and $n = 1$, and $\frac{1}{C} = \frac{1}{2} \cdot \frac{\alpha \gamma}{m}$. As the parameter of the parabola (C) can not be determined from the above equation because (m) is still unknown, it must therefore be determined experimentally, — which Humphreys and Abbott also essayed to accomplish. In their experiments on the Mississippi, they discovered that this parameter (C) varies inversely as the square root of another quantity C and as another experimental coefficient K, and directly as the square of the depth a to the bottom, and thus established the following formula:

$$C = \frac{a^2}{K\sqrt{K.c}}$$

By substituting this value of (C) in the above formula, we obtain:

$$v = v' - K\sqrt{K.c}\left(\frac{x - a_0}{a}\right)^2 \quad \ldots\ldots \quad (1)$$

The value of (K) as found by Humphreys and Abbott, when reduced to the metrical system, is the following:

$$K = \frac{0{,}284}{\sqrt{\frac{c}{6}} + 0{,}46}.$$

This value of the co-efficient (K) is, from the nature of the case, reliable only for large streams like the Mississippi. H. & A. also found that the ratio of the depth (a_0) to the entire depth (a) was quite great, i.e., that the axis of the parabola lies quite deep below the surface of the water. They express this ratio thus:

$$\frac{a_0}{a} = 0{,}317 + 0{,}06\,f,$$

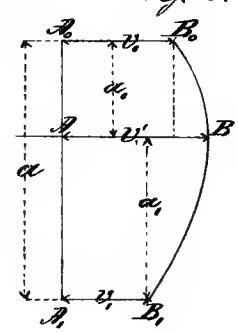
where (f) is a certain magnitude or measure depending upon the direction and velocity of the wind, the direction being considered

positive when blowing up-stream, and negative in the contrary direction. For a hurricane blowing up stream (+), the value of (f) is: $f = 10$.

Too great reliance must not be placed in this formula, as the definition of (f) is still very uncertain. The effect of this action of the wind, is to lower the axis of the parabola when blowing up-stream, and to raise it when blowing down-stream, or in other words, it causes an acceleration or a retardation of the velocity of the water at its surface.

The exact reason why the velocity of the water diminishes from the axis of the parabola downwards, is not clearly known. It is presumed, however, that the roughness of the bed has much to do with this; and if we also can assume that upward currents are produced by the impact of the water against obstacles, such as stones, etc.; or in general, that a mingling of the lower with the upper strata of the body of flowing water actually takes place, we can then conceive how these causes, along with the constant friction upon the bed, may tend to produce a diminution in the velocity from the axis down-wards.

Fig. 8.



The calculation of the mean velocity (v) is performed, when several other velocities in the same vertical have been observed and noted, by remembering that the separate velocities may be represented graphically by areas, and hence the mean velocity by an area (v.a) which is equal to the sum of the areas of the other velocities. Assuming here that the velocities v_0, v', and v_1, Fig. 8, are known, as well as their location relatively to the surface, we then have for the mean velocity: (See Weisbach, Vol I p. 101.)

$$v = \left\{ a_0 v_0 + \tfrac{2}{3} a_0 (v' - v_0) + a_1 v_1 + \tfrac{2}{3} a_1 (v' - v_1) \right\} \cdot \frac{1}{a},$$

and reducing:

$$v = \tfrac{2}{3} v' + \tfrac{1}{3} \cdot \frac{a_0}{a} \cdot v_0 + \tfrac{1}{3} \cdot \frac{a_1}{a} \cdot v_1 .$$

But we have already found for the surface velocity (v_0):

$$v_0 = v' - \sqrt{Rc} \cdot \left(\frac{a_0}{a}\right)^2,$$

and by making $x = a$ in formula (I) page 7, we have for the velocity upon the bed:

$$v_1 = v' - \sqrt{Rc} \cdot \left(\frac{a_1}{a}\right)^2$$

Substituting these values of (v_0) and (v_1) in the above equation for the mean velocity, we obtain:

$$\bar{u} = v' - \frac{\sqrt{K.C}}{3}\left[\left(\frac{a_0}{a}\right)^3 + \left(\frac{a_1}{a}\right)^3\right].$$

Accordingly the mean velocity would now be known, if we only knew the exact position of the point of greatest velocity in the vertical line. We must therefore endeavor to get rid of the term involving (a_0). Developing the value of (u), we obtain; by substituting for (a_1) its value $=(a - a_0)$:

$$u = v' - \frac{\sqrt{K.C}}{3}.\frac{a^3 - 3a^2a_0 + 3aa_0^2}{a^3} = v' - \frac{\sqrt{K.C}}{3}.\frac{a^2 - 3aa_0 + 3a_0^2}{a^2};$$

and by introducing the value of $(v') = v + \sqrt{KC}\left(\frac{\kappa - a_0}{a}\right)^2$ from formula (1), p. 7, we have:

$$u = v - \frac{\sqrt{KC}}{3}.\frac{a^2 - 3aa_0 + 3a_0^2 - 3(\kappa - a_0)^2}{a^2} = v - \frac{\sqrt{KC}}{3}.\frac{a^2 - 3[a_0(a - 2\kappa) + \kappa^2]}{a^2}.$$

Fortunately there is a point where, for $\kappa = \frac{a}{2}$, the term containing (a_0) becomes $= 0$, and the fraction of which it formed a part reduces itself to $(\frac{1}{4})$; and hence we have finally:

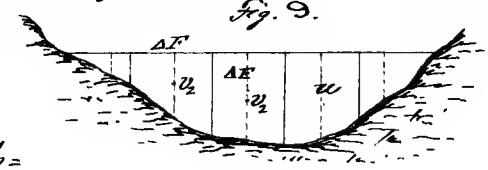
$$u = v_2 - \frac{\sqrt{K.C}}{12}, \qquad \text{(I.)}$$

where (v_2) denotes that value of the general (v) which corresponds to the depth $\kappa = \frac{a}{2}$.

Therefore to ascertain the mean velocity in any vertical line, it is necessary to know and measure the velocity at ½ the entire depth of the water at that vertical.

If it is required to find the mean velocity (c) of the entire cross-section, the latter is divided up into bands, as in Fig. 9., and the velocities (v_2) at one-half the depth of the verticals which pass thro' the centres of gravity of these bands, measured; then, by means of Equation (I.) above, the mean velocity of each band may readily be found, and then the mean velocity of the entire cross-section can be obtained.

Fig. 9.



The quantity of water flowing thro' this cross-section may be placed:

$$Q = F.c' = \Sigma(u.\Delta F) = \Sigma(v_2 \Delta F) - F.\frac{\sqrt{KC}}{12}.$$

Transposing, we have:

$$c + \frac{\sqrt{K}}{12}.\sqrt{C} = \frac{\Sigma(v_2.\Delta F)}{F} = C_2,$$

where the entire second member of the Equation is denoted by (C_2).

Solving the equation, we find:

$$\sqrt{C} = -\frac{\sqrt{K}}{24} + \sqrt{\left(\frac{\sqrt{K}}{24}\right)^2 + C_2}, \text{ or, squaring: } C = \left[\sqrt{C_2 + \frac{K}{576}} - \sqrt{\frac{K}{576}}\right]^2.$$

If all the values of (v_2) can not be observed by means of a float and a large body suspended in the water at the proper depths below it, as many as possible may be observed, and the remaining ones interpolated.

We must now ascertain the relations of the mean velocity (c) of the entire cross-section to the relative fall and the dimensions of this cross-section. For this purpose, let us assume that the water between any two cross-sections, whose distance apart is (dx), is in a state of uniform permanency (gleichförmiger Beharr. Zustand).

Let:- (α) = the relative fall;
(F) = the area of the cross-section;
(p) = the wetted cross-profile, and
$\gamma . F . dx . \alpha$ = the moving force

This latter, in order to fulfill the condition of uniform permanency, must be equal to the resistances encountered, - that is:

$$\gamma . F . dx . \alpha = dx . p = dx \int dp . f(v_1),$$

where (v_1) denotes the velocity of the water along its bed.

As this formula is somewhat complicated, it is preferable to write:

$$\gamma . F . dx . \alpha = dx . \int dp . f(v_1) = dx . p . f(c),$$

whence:

$$\frac{F}{p} . \alpha = \frac{1}{\gamma} . f(c) = \varphi(c), \text{ since } \frac{1}{\gamma} \text{ is constant.}$$

Further, let $\frac{c^2}{2g}$ = the living force per kilogramme of water, and

" - $\xi . \frac{c^2}{2g}$ = the loss in living force per kilog. of water occasioned by passing over the length (dx) of channel.

According to d'Aubuat's principle, we may then place:

$$\xi . \frac{c^2}{2g} = \alpha . dx ; \text{ or we may place the co-efficient of resistance:}$$

$$\xi = \gamma . dx . \frac{p}{F} .$$

Substituting this value in the last equation, and then multiplying both members by $(\frac{F}{p})$, we obtain:

$$\gamma . \frac{c^2}{2g} = \frac{F}{p} . \alpha = \varphi(c) .$$

The older writers upon Hydraulics placed: $\varphi(c) = Ac + Bc^2$, A and B being constants to be determined by experiments.

Prony found the following values for A and B:

$$A = 0{,}0000\text{xx} \quad \text{and} \quad B = 0{,}000309$$

Eytelwein:

" = 0,000024 " " 0,000306

Lamaire

" 0,000022 " " 0,000378.

Mean Values: " 0,000030 " " 0,000351

We may therefore place:

$$\varphi(c) = Ac + Bc^2 = \frac{c^2}{K^2},$$

understanding by $\frac{1}{K^2}$ the expression: $\frac{A}{c} + B = \frac{1}{K^2}$; whence:

$$\frac{F}{p}.\alpha = \frac{c^2}{K^2}, \quad \text{and} \quad c = K.\sqrt{\frac{F}{p}.\alpha}.$$

The following values of c and K, were found experimentally by

For: $c = 0{,}5$ | $= 1{,}0$ | $= 1{,}5$ | $= 2{,}0$ |
$K = 49{,}3$ | $= 51{,}2$ | $= 51{,}9$ | $= 52{,}3$ |

We thus see that while (c) varies considerably, the value of (K) on the other hand, fluctuates between comparatively small limits.

The values of these constants were thoroughly ascertained in the course of the experiments instituted by the French Government under the direction of Darcy, and after his death, continued under that of Bazin. — From the results obtained, it was found that:

$$c = \sqrt{\frac{\frac{F}{p}.\alpha}{\alpha + b.\frac{p}{F}}} = K.\sqrt{\frac{F}{p}.\alpha},$$

where (K) has an entirely different signification from that which was assigned to it in the other formula above. Here its value is:

$$\frac{1}{K^2} = \alpha + b.\frac{p}{F},$$

where (α) and (b) denote coefficients depending upon the state and nature of the channel's surface. The coefficient (K) is thus strictly not constant, and depends not so much upon the mean velocity as upon the state of the bed and sides of the canal.

From the experiments just mentioned, it was found that when the channel was quite smooth, as of cement or planed planks, the values of (α) and (b) were:

(1)..... $\alpha = 0{,}00015$, and $b = 0{,}0000045$;

for a channel with a rougher surface, such as brick, etc:

(2)..... $\alpha = 0{,}00019$, and $b = 0{,}0000124$;

when the channel was built of rough quarry stones:

(3)..... $\alpha = 0{,}00024$, and $b = 0{,}0000600$;

and finally, when the channel was constructed from earth only:

(4)..... $\alpha = 0{,}00028$, and $b = 0{,}0003500$

According to the experiments of Humphreys and Abbott, we have, on the other hand:

$$(0{,}93.C + 1/6\sqrt{K}.C)^2 = 59{,}4.\frac{F}{b+p}.\sqrt{\alpha},$$

whence:

$$K = \frac{0{,}284}{\sqrt{\frac{F}{b}} + 0{,}46}.$$

This formula is expressed in the metrical system, and is purely empirical.

In order to simplify the calculation of the mean velocity, we may, however, place:

$$c = m.\sqrt{\frac{F}{b+p}}.\sqrt{\alpha},$$

where the co-efficient:

$$m = \frac{\sqrt{59.4}}{0,93 + \frac{1}{6}\sqrt{\frac{F}{c}}}.$$

In the Remark to Art. 156, Redtenbacher's Resultaten, in the Appendix by Prof. Grashof, will be found a table calculated from this formula, for the values of the mean velocity. These latter are taken within the limits: $c = 0,5^{m}$ and $2,5^{m}$, while the mean depth of the water ranges from: $\frac{F}{b} = 0,25^{m}$ to $= 16.0^{m}$. Within these limits, it is found that (m) varies:

from: ---- $m = 7,25$ to $m = 8,08$

When the water is in a variable state of permanency, it also becomes necessary to understand the relations of its motion or flow. For example, when a stream or brook is dammed up for the purpose of feeding a hydraulic motor, the water above the dam will then be in this state. Accordingly, we will denote by (v) the mean velocity of the water in any cross-section, by F, the area of the latter, and by Q the quantity of water flowing thro' it.

We have thus the three quantities Q, F, and (v) to consider, of which Q alone is constant, while F and (v) may vary, though their product $(F.v)$ is constant; hence:

$$Q = F.v.$$

Next, let $(d\eta)$ be the fall for an infinitely small distance (dx) in which the water has been assumed to be in a state of variable permanency. The relative fall is therefore $\left(\frac{d\eta}{dx}\right)$ and will be denoted by (φ); that is,

$$\varphi = \frac{d\eta}{dx}.$$

Conceive this infinitely thin vertical lamina of water, having the velocity (v) at its upper end-surface and $(v + dv)$ at its lower end-surface whose distance from the first is (dx).

If (m) = the mass of this lamina of water, then:

$\frac{m.v^2}{2}$ = its living force at its upper surface (relative of course to the direction of the current) and:

$\frac{d\,m.v^2}{2}$ = the increase of living force produced in passing thro' the distance (dx).

Now if the water was in a uniform state of permanency, we should have:

$$\frac{d\,m.v^2}{2} = m.g.\varphi$$

But all of this increment of living force is not applied to augmenting the entire living force of the lamina of water. A portion of it must be used in overcoming the frictional resistances; therefore this relative fall may be conceived as consisting of two parts, and hence

$$\frac{d\,m.v^2}{2} = mg\left(\varphi - \frac{p.v^2}{F.K^2}\right) dx .$$

Since $\frac{d\,m.v^2}{2} = m.v\,dv$, we may divide the equation by (m) and thus obtain, remembering that $\varphi = \frac{d\eta}{dx}$:

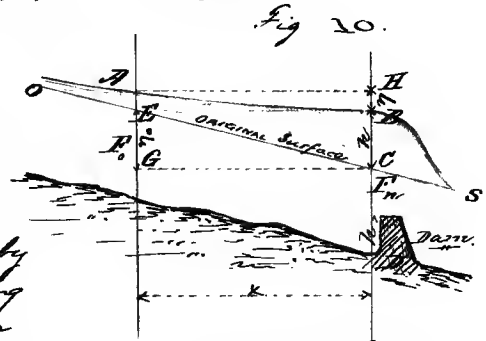
$$d\eta = \frac{v\,dv}{g} + \frac{p}{F}\cdot\frac{v^2}{K^2}\,dx, \qquad \text{---------- (1)}$$

which is the fundamental formula for the motion of water in a variable state of permanency (ungleichförmiger Beharrungs Zustand.)

Integrating, we get:

$$\eta = \frac{v_m^2 - v_0^2}{2g} + \int_0^{x_m} \frac{p}{F}\cdot\frac{v^2}{K^2}\cdot dx \qquad \text{-----(2.)}$$

Fig 10.



For example, in Fig. 10., let OS be the original water surface, and AB, that produced by the Dam D. Required to find the fall (η) between any two given points. Let us here denote by

Q, the quantity of water passing thro the cross-section F_p per second; and because $v = \frac{Q}{F}$, we accordingly have from the last formula:

$$\eta = \frac{Q^2}{2g}\left(\frac{1}{F_m^2} - \frac{1}{F_0^2}\right) + Q^2\int_0^{x_m} \frac{1}{K^2}\cdot\frac{p}{F^3}\cdot dx .$$

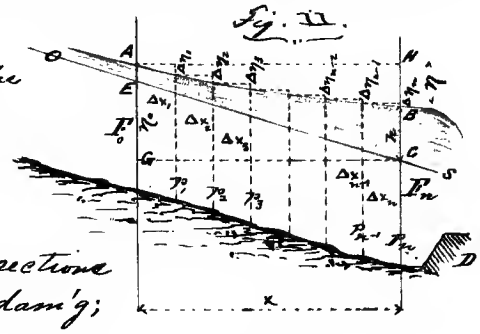
We are thus to determine, — when a dam of a certain height (h) is to be built at some given point in the stream, — how much the original water surface at any given point above the projected dam will thereby be raised. Accordingly:

denote by (η_0) — the original fall EG, and
" (η) — the new (unknown) fall HB; and hence
$(h + \eta - \eta_0)$ is the elevation AE above the original surface at A. In the problem (h) and (η_0) are supposed to be given, while (η) is to be calculated from the formulas.

In order to accomplish this, the quantity of water per second, as well as the nature and configuration of the channel by means of any number of measurements, soundings and Profile, must thoroughly be known. The distance between A and B is then divided into a number of sufficiently small portions or bands by vertical planes, so that we may substitute constant mean values, without sensible error, for the velocities

between any two successive cross-sections, as well as such values for the wetted cross-profiles (p) and the cross-sections (F) themselves, as in Fig. II.

Fig. II.



In Fig. II, let us denote as before:

$p_0, p_1, p_2 \ldots p_n$ = the wetted cross profiles at the respective (n) sections;

η_0 = the fall between the points A & B before damming;

η = the fall after damming;

$F_0, F_1, F_2, \ldots F_n$ = the areas of the cross-sections at the (n) vertical sections, after dam'g;

h = the height of the dam D;

$\Delta\eta_1, \Delta\eta_2, \Delta\eta_3 \ldots \Delta\eta_n$ = the fall after dam'g between the two respective successive cross-sections F_0 & F_1; $\ldots F_{n-1}$ & F_n;

$\Delta x_1, \Delta x_2, \Delta x_3 \ldots \Delta x_n$ = the distances between these cross-sections, and

x = distance between A and B measured horizontally.

Of all these quantities, only F_n, p_n, x, h, Δx, and η_0 are known, and it is now the problem to find approximately the value of (η).

For starting, we must assume some value for ($\Delta\eta_n$) and since its value is least at the point B, we will accordingly begin at that point and place approximately $\Delta\eta_n = 0$. On this assumption, the values of F_{n-1} and p_{n-1} may then be calculated from the above formulas, and by substituting them thus found in the formula for the value of ($\Delta\eta_n$), we obtain a corrected value for the latter:

$$\eta = \Delta\eta_n = \frac{Q^2}{2g}\left(\frac{1}{F_n^2} - \frac{1}{F_{n-1}^2}\right) + \frac{Q^2}{K^2}\left(\frac{p_n}{F_n^3} + \frac{p_{n-1}}{F_{n-1}^3}\right)\frac{\Delta x_n}{2}.$$

We must next assume some value for ($\Delta\eta_{n-1}$), which for the present is placed equal to the value last found for ($\Delta\eta_n$). The values of F_{n-2} and p_{n-2} are then calculated as above, and by their substitution in the formula for ($\eta = \Delta\eta_{n-1}$), a corrected value of $\Delta\eta_{n-1}$ is finally found, viz:

$$\eta = \Delta\eta_{n-1} = \frac{Q^2}{2g}\left(\frac{1}{F_{n-1}^2} - \frac{1}{F_{n-2}^2}\right) + \frac{Q^2}{K^2}\left(\frac{p_{n-1}}{F_{n-1}^3} - \frac{p_{n-2}}{F_{n-2}^3}\right)\frac{\Delta x_{n-1}}{2}.$$

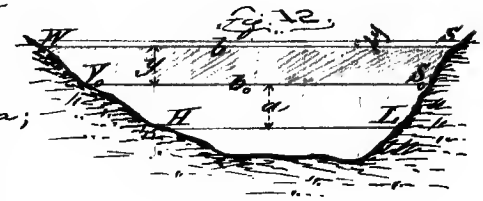
Proceeding in this manner for the remaining sections, we at length obtain:

$$\eta = \Delta\eta_1 + \Delta\eta_2 + \Delta\eta_3 + \cdots + \Delta\eta_{n-1} + \Delta\eta_n.$$

This method is however so tedious, that it becomes necessary to seek after some simpler plan. Let us therefore assume that the bed of the stream is a cylindrical surface, the axis of which is inclined to the horizon under a small angle (α). Further, the water is in a state of uniform permanency, the surface of which is also inclined to the horizon under precisely the same angle (α)

The form of the cross-section may nevertheless be arbitrary. Let us assume such an one as is shown in Fig 12.,

$V_0 S_0$ = the original surface of the water;
p_0 = the original wetted surface;
F_0 = " " " Cross-sectional area;
a = " mean depth of the water,
and: b_0 = " breadth at the surface of the water



We thus have: $a = \frac{F_0}{b_0}$.

Next conceive a horizontal line HI to be drawn at a depth $= (a)$ below $V_0 S_0$. This line is called the Base line (grund-linie) and the horizontal plane passing thro it the Base plane (Ground plane).

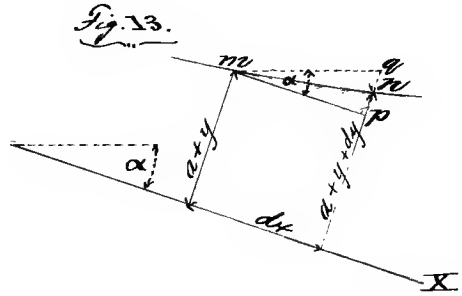
Also let WS be the surface of the water after damming, and its height above $V_0 S_0$, be $= y$. Considering the area of the cross-section to be known, we accordingly have:

b, p, F, Q, and $v = \frac{Q}{F}$, which are thus all known functions.

Let the inclined longitudinal section of the water be taken as the axis of X, whose ordinates (y) may be measured indifferently either in a vertical direction, or in one normal to this axis.

In Fig. 13., let (m) and (n) be two consecutive points in the "swell curve" (Stau-Curve) whose distance apart measured on a parallel to the axis of X is (dx); and let (α) be the inclination of this axis to the horizon; and then draw (mp) parallel to X, and (mq) horizontal.

Fig. 13.



We thus have here: $pn = dy$;
$nq = d\eta$; and
$pq = \alpha . dx$, sensibly;
$d\eta = \alpha . dx - dy$, and $(F.v) =$ constant.

Differentiating $F.v =$ Const., on the supposition that both (F) and (v) may vary, we obtain:

$$F.dv + v.dF = 0, \text{ whence}$$
$$dv = -\frac{v.dF}{F} = -\frac{b}{F}.v.dy,$$

where (dF) may be replaced by its value $(b.dy)$, (b) being as noted above, the breadth of the raised water surface, and (dy) the height of an elementary horizontal area or band, as in Fig. 12. Substituting these values of $(d\eta)$ and (dv) in the equation for the value of $(d\eta)$, on page 13., eq. (2.),

$$d\eta = \frac{v.dv}{g} + \frac{p}{F} \cdot \frac{v^2}{K^2} \cdot dx,$$

we shall obtain:

$$\alpha . dx - dy = -\frac{b}{F} \cdot \frac{v^2}{g} \cdot dy + \frac{p}{F} \cdot \frac{v^2}{K^2} \cdot dx ,$$

and by transposition:

$$dx = \frac{1 - \frac{b}{F} \cdot \frac{v^2}{g}}{\alpha - \frac{p}{F} \cdot \frac{v^2}{K^2}} \cdot dy . \quad \text{---------- (1.)}$$

The entire second member of this equation is a function of (y), and we may therefore place:

$$dx = Y . dy ,$$

whence by integration there follows:

$$x = \int_y^{y_n} Y . dy$$

The expression (Y) is however such a function of (y) as can not be completely integrated by means of the ordinary rules. We may therefore apply with advantage Simpson's Rule, if the difference between (y_n) and (y) is divided into an even number of equal parts. Hence placing:

$$\begin{aligned} y &= Y \\ y + \Delta y &= Y_1 \\ y + 2\Delta y &= Y_2 \\ \cdots\cdots & \quad . \\ \cdots\cdots & \quad . \\ y_n \cdots &= Y_n , \end{aligned}$$

we obtain finally:

$$x = \frac{Y_n - Y}{3n} \left(Y + 4Y_1 + 2Y_2 + \cdots\cdots + 4Y_{n-1} + Y_n \right) .$$

Making the case a still more special one, we will assume that the breadth (b) is very great in comparison to the depth $(a + y)$, where (a) denotes the mean depth of the water before damming, and (y) the amount which this original water surface is to be raised. Furthermore, assume that the breadth (b) is such that a constant mean value can be assigned to it in the calculations, which would be the case if the banks of the stream above the original surface of the water were vertical. The steeper the banks of the stream, therefore, the nearer may we use one constant mean value for all the cross-sections; and in the following case, we will assume that they are sufficiently steep as to allow our using this constant mean value (b) for the breadth.

Denoting the original cross-section by F_0, we then have for the cross-section after the surface is raised:

$$F = F_0 + by ;$$

and:

$$\frac{F}{b} = a + y .$$

The formula for (dx). — eq. (2). page. 26., — then becomes, if we remember that $v = \frac{Q}{F}$, and assume that we may replace (p), the wetted cross profile, by the mean breadth (b):

$$dx = \frac{1 - \frac{1}{2}\cdot\frac{Q^2}{b^2}\cdot\frac{b^3}{F^3}}{\alpha - \frac{1}{K^2}\cdot\frac{Q^2}{b^2}\cdot\frac{b^3}{F^3}}\cdot dy = \frac{1 - \frac{1}{2}\cdot\frac{Q^2}{b^2}\cdot\frac{1}{(a+y)^3}}{\alpha - \frac{1}{K^2}\cdot\frac{Q^2}{b^2}\cdot\frac{1}{(a+y)^3}}\cdot dy.$$

We here omit, however, that (k) and (K) are not constant and it may thus be replaced, since it is so extremely insignificant.

For the original state of uniform permanency of the water, before damming, we have the expression for its mean velocity:

$$c = \frac{Q}{ab} = K\sqrt{\frac{ab}{b}}\cdot\alpha = K\sqrt{\alpha}\,\alpha,$$

where, as above, we place $F = ab$ and $p = b$. Squaring the last equation, we have:

$\frac{Q^2}{b^2} = K^2 a^3 \alpha$, and introducing this value of $\frac{Q^2}{b^2}$ into the above differential equation, we obtain:

$$dx = \frac{1 - \frac{\alpha K^2}{2}\left(\frac{a}{a+y}\right)^3}{\alpha - \alpha\left(\frac{a}{a+y}\right)^3}\cdot dy.$$

Let us denote the ratio $\left(\frac{a+y}{a}\right)$ by (z), i.e., place $a+y = az$, where (z) denotes the ratio of the mean depth of the water originally, to the mean depth of the stream after its surface is raised. Hence:

$dy = a\cdot dz$,

and with this substitution, we shall obtain:

$$dx = \frac{a}{\alpha}\cdot\frac{z^3 - \frac{\alpha K^2}{2}}{z^3 - 1}\cdot dz.$$

This fractional algebraic rational function of (z) may be resolved into regular functions of (z); that is, we may place:

$$dx = \frac{a}{\alpha}\left(1 - \frac{1 - \frac{\alpha K^2}{2}}{z^3 - 1}\right)\cdot dz$$

Now let (h) be the height of swell (Stauhöhe) immediately above the dam: and let $(\alpha + h) = az_1$, where (z_1) denotes the maximum value of the variable (z). Accordingly, in order to find the distance (x) from the dam to a point in the stream where its surface is raised an amount $= (z)$, this last equation must be integrated between the limits (z) and (z_1). Performing this operation, we get:

$$x = \frac{a}{\alpha}\left[z_1 - z - \left(1 - \frac{\alpha K^2}{2}\right)\cdot\int_z^{z_1}\frac{dz}{z^3 - 1}\right].$$

Let us denote the indefinite integral $\left(-\int\frac{dz}{z^3-1}\right)$ by $\psi(z)$; then integrating between the limits (z) and (z_1), we have:

$$x = \frac{a}{\alpha}\left[z_1 - z + \left(1 - \frac{\alpha K^2}{2}\right)\left(\psi(z) - \psi(z_1)\right)\right] \quad \text{-----} (1)$$

The value of (x) would now be known if we only knew that of $\psi(\zeta)$. This we must accordingly endeavor to do. For this purpose, resolve the expression: $\left(\frac{d\zeta}{\zeta^3-1}\right)$ into two factors, viz:

$$\frac{d\zeta}{\zeta^3-1} = \frac{1}{3}\left(\frac{1}{\zeta-1} - \frac{\zeta+2}{\zeta^2+\zeta+1}\right) d\zeta,$$

which may in turn again be separated into:

$$\frac{d\zeta}{\zeta^3-1} = \frac{1}{3}\left(\frac{d\zeta}{\zeta-1}\right) - \frac{1}{6}\left(\frac{2\zeta+1}{\zeta^2+\zeta+1}\cdot d\zeta\right) - \frac{1}{2}\left(\frac{d\zeta}{\frac{3}{4}+(\zeta+\frac{1}{2})^2}\right);$$

and since $\int\frac{d\zeta}{\zeta-1} = l.n(\zeta-1)$, etc., we have again:

$$\frac{d\zeta}{\zeta^3-1} = \tfrac{1}{3}\cdot d.l.n(\zeta-1) - \tfrac{1}{6}\cdot d.l.n(\zeta^2+\zeta+1) - \tfrac{2}{3}\cdot\left(\frac{d\zeta}{1+\frac{(2\zeta+1)^2}{3}}\right)$$

$$= \tfrac{1}{3}.d.l.n(\zeta-1) - \tfrac{1}{6}.d.l.n(\zeta^2+\zeta+1) - \frac{1}{\sqrt{3}}\cdot d.arc.tang.\left(\frac{2\zeta+1}{\sqrt{3}}\right),$$

remembering that: $d.arc.tang.\,x = \frac{dx}{1+x^2}$.

The integration may now be completed, obtaining thereby:

$$\psi(\zeta) = -\tfrac{1}{3}.l.n(\zeta-1) + \tfrac{1}{6}\cdot l.n(\zeta^2+\zeta+1) + \frac{1}{\sqrt{3}}\cdot arc.tang.\left(\frac{2\zeta+1}{\sqrt{3}}\right) + Constant,$$

$$= \tfrac{1}{6}\cdot l.n\left\{\frac{\zeta^2+\zeta+1}{(\zeta-1)^2}\right\} + \frac{1}{\sqrt{3}}\,arc.tang.\left(\frac{2\zeta+1}{\sqrt{3}}\right) + Const.$$

Let us take this constant so that when $\zeta=\infty$ —

We then have:

$$\psi(\infty) = \frac{1}{\sqrt{3}}\cdot\frac{\pi}{2} + Const. = 0\ ;$$

on this supposition, therefore, if $(\zeta)=\infty$, the constant is $= -\frac{1}{\sqrt{3}}\cdot\frac{\pi}{2}$, and hence:

$$\psi(\zeta) = \frac{1}{6}\cdot l.n\left\{\frac{\zeta^2+\zeta+1}{(\zeta-1)^2}\right\} - \frac{1}{\sqrt{3}}\cdot arc.tang.\left(\frac{2\zeta+1}{\sqrt{3}}\right) - \frac{1}{\sqrt{3}}\cdot\frac{\pi}{2} \qquad (2)$$

Therefore, when (ζ) and (ζ_1) are given, the value of (x) may now be found from the above equation (1) p. XI.

To find where the raised water surface runs into the original surface of the stream, we must place $y=0$ or $\zeta=1$. On this supposition, the first term of eq. (2) for the value $\psi(\zeta)$ becomes $=\infty$, while the second term, however, is finite. Theoretically, therefore, the raised or curbed water surface bears the relation of an asymptote to the original surface, or that the effect of the dam is theoretically felt at an almost infinite distance from its location.

For the practical use of the above formula for the value of $\psi(x)$ it becomes extremely convenient to have table giving the values of $\psi(z)$. Such a table will be found in Redtenbacher's Resultaten, Appendix, under the Remark to № 149, and is arranged for increasing values of $(\frac{1}{z})$ that is from $\frac{1}{z} = 1 - \frac{1}{\infty}$ to $z = 1 - z = \infty$

Example. In a certain stream, we have given:
the breadth $(B) = 20.^{m}$;
the original cross-section: $F_0 = 15^{\text{sq.m}}$;
the " " mean depth of water: $(\alpha) = \frac{F_0}{B} = 0{,}75^{m}$;
" wetted cross-profile: $p = 21^{m}$;
" quantity of water: $Q = 12$ cubic metres per second; and hence in a state of uniform permanency, the mean velocity:

$$C = \frac{Q}{F_0} = \frac{12}{15} = \frac{4}{5} = 0{,}8^{m} \text{ per second.}$$

A dam is to be built across this stream of such a height as will cause the water surface to be raised 1^{m}. Required the distance from the dam to the point where the swell will be $0{,}1^{m}$. For the solution of this problem, the values of (α) and (k) must also be known. Although the former might be found by means of a preceding formula, yet it is such a simple matter to obtain it by direct levelling, that it is generally found in this manner. Assume that the value thus obtained is $(\alpha) = 0{,}0006$; and we therefore have:

$$\frac{\alpha_1}{\alpha} = \frac{0{,}75}{0{,}0006} = 1250.$$

In the original condition of the stream the value of C is found from the formula

$$c = 0{,}8 = k\sqrt{\tfrac{15}{21}} \cdot \alpha$$

whence

$$\frac{\alpha k^2}{g} = \frac{0{,}64 \cdot 21}{15 \cdot 9{,}81} = 0{,}0913$$

Further: $\alpha + h = 1 + 0{,}75 = 1{,}75$, and therefore:

$$z_1 = \frac{1{,}75}{0{,}75} = 2{,}333,$$

or, as the values in the table are arranged for the reciprocal of (z):

$$\frac{1}{z_1} = 0{,}429.$$

By interpolation between the two nearest values in the table, we now find: $\psi(z_1) = 0{,}0950$

Again:

$$\alpha + y = 0{,}85, \quad z = \frac{0{,}85}{0{,}75} = 1{,}133, \text{ and as before by}$$

interpolation: $\psi(z) = 0{,}5940$,
whence: $\psi(z) - \psi(z_1) = 0{,}499$,
and:

$$x = 1250\,(1{,}2 + 0{,}9087 \cdot 0{,}499) = 2066 \text{ metres.}$$

Were this raising of the original water surface of 0,1 m. not allowable at this distance of 2066 m., the height of the dam would therefore have to be diminished.

At different seasons of the year, the water in streams rises or falls to various levels, and it thus becomes interesting and useful to understand the relations of its flow when in these various states. In a condition of rapid increase or sudden falling of the levels, the relations of its flow assume very complex forms; and in the following examination, the water will therefore be considered as being in a state of uniform permanency, and not in a transition state. Furthermore an exordinately great increase of the quantity of water (Q) will not be assumed.

We will also take it for granted that for a certain state of uniform permanency, the values of (α), (C), (Q), (F), (p), (k) and (α) to be known. The two last, (α) and (k), may be considered as constant, since their mean values are not sensibly affected by such changes in level as presupposed. The value of (F) and with it, (p), will however vary.

For another state of uniform permanency, we thus have the mean velocity:

$$C + \Delta C = k\sqrt{\frac{F+\Delta F}{p+\Delta p}} \cdot \sqrt{\alpha},$$

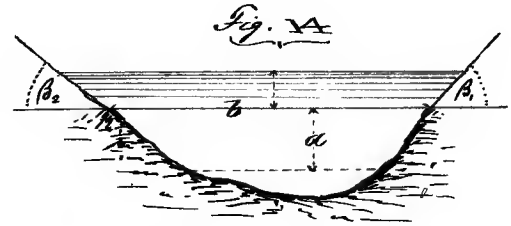
and by separating the terms under the radical sign, this factor becomes

$$k\sqrt{\frac{F+\Delta F}{p+\Delta p}} \cdot \sqrt{\alpha} = C\sqrt{\frac{1+\frac{\Delta F}{F}}{1+\frac{\Delta p}{p}}},$$

whence:

$$\frac{\Delta C}{C} = \sqrt{\frac{1+\frac{\Delta F}{F}}{1+\frac{\Delta p}{p}}} - 1.$$

Fig. 14



Let us assume any cross-section, as in Fig. 14, and assume that the rise or difference in level between the two states of the stream is to be known by observation. We may therefore find (ΔF) and (Δp) by calculation, and then also (ΔC). Again:

$$Q + \Delta Q = (F + \Delta F)\cdot(C + \Delta C) = Q\left(1+\frac{\Delta F}{F}\right)\left(1+\frac{\Delta C}{C}\right), \text{ whence}$$

by substituting the value found above for $\left(\frac{\Delta C}{C}\right)$:

$$\frac{\Delta Q}{Q} = \frac{\left(1+\frac{\Delta F}{F}\right)^{3/2}}{\sqrt{1+\frac{\Delta p}{p}}} - 1.$$

From this equation, the difference in the quantity of water in the two states may now be found.

Let the inclinations of the banks of the stream above the first state of the water's surface be such that for the angles (β_1) and (β_2)

of the slopes, we may substitute a mean value (β). Further, let (a) denote the mean depth of the water in the first state, and the area of its cross-section be: $F = ab$; also let (Δa) be the difference in level between the two given states, which is known.

We may then place:

$$\Delta F = b \cdot \Delta a + (\Delta a)^2 \cdot \operatorname{colg.}\beta ,$$

where the last term expresses the area of two right-angled triangles which go to make up the trapezoidal figure of ΔF, a mean value having been taken for the two angles (β_1) and (β_2) equal to their arithmetical mean. Dividing this equation by the equation $F = ab$, we shall obtain:

$$\frac{\Delta F}{F} = \frac{\Delta a}{a}\left(1 + \frac{\Delta a}{b} \cdot \operatorname{cotang.}\beta\right).$$

With reference to the ratio $\left(\frac{\Delta p}{p}\right)$, we have:

$$\frac{\Delta p}{p} = \frac{2 \cdot \Delta a}{p \cdot \sin.\beta} .$$

These values may now be substituted in the formula for $\left(\frac{\Delta c}{c}\right)$ and then into that for the value of $\frac{\Delta Q}{Q}$.

Assume again that the angle (β) is not very small and hence also that ($\sin.\beta$) is not very large; furthermore that: $\left(\frac{\Delta a}{d}\right)$ and $\left(\frac{a}{b}\right)$ are only small fractions.

Considering only such quantities as are not less than very small magnitudes of the second order, — such as $\left(\frac{\Delta p}{p}\right)$ for instance — — and neglecting those of the third order, we will thus obtain:

$$\frac{\Delta c}{c} = \left(1 - \frac{\Delta a}{p \sin.\beta}\right)\sqrt{1 + \frac{\Delta a}{a}} - 1 = \left(1 + \frac{\Delta a}{p \sin.\beta}\right)\left(1 + \frac{1}{2} \cdot \frac{\Delta a}{a} - \frac{1}{8}\left(\frac{\Delta a}{a}\right)^2\right) =$$

$$= \frac{1}{2} \cdot \frac{\Delta a}{a}\left(1 - \frac{1}{4} \cdot \frac{\Delta a}{a}\right) - \frac{\Delta a}{p \cdot \sin.\beta} .$$

In a precisely similar manner, we finally obtain for $\left(\frac{\Delta Q}{Q}\right)$ the value:

$$\frac{\Delta Q}{Q} = \frac{2}{3} \cdot \frac{\Delta a}{a}\left(1 + \frac{1}{4} \cdot \frac{\Delta a}{a}\right) - \frac{\Delta a}{p \cdot \sin.\beta} .$$

— " — 9 — " —

DAMS.

Dams are built for the purpose of concentrating the fall of a stream, and are distinguished into *Overfall*, *partial Overfall* and *Sluice Dams* or *Weirs*.

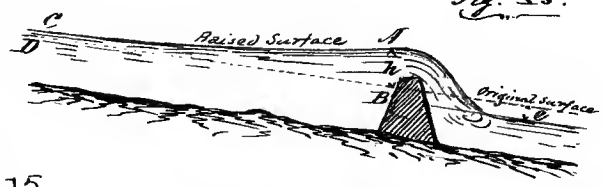
When the crown of the dam is higher than the surface of the water below it, it is called an *Overfall Dam*; but when the crown lies below the surface of the water just beyond, it receives the name of a *partial overfall Dam*; and lastly, when the weir is provided with openings which may be regulated by means of gates, slē. the structure is called a *Sluice Dam*.

Only a portion of the entire amount of water carried down by the stream flows over the weirs, since the rest is conducted away through the upper race to supply the hydraulic motor, on whose account the dam was built. Sluice Dams are generally adopted only when the fluctuations in the level of the stream are very great.

In laying out a dam, it is of great importance to know how far above the dam the water will be swelled (Stau write) and how much, at a given distance and with a given height of the dam, the original surface of the water will be raised. This height is called the "*height of the swell*" (Stauhöhe).

This last is always estimated from the original surface of the stream, and at the dam itself, it is measured at such a distance above, where the water surface has not yet commenced to curve in flowing over the weir. In Fig. 15, *AB* is the height of the swell, and will hereafter be denoted by (*h*).

Fig. 15.



In consequence of a portion of the water being carried away by the upper race or feeder, the original surface of the stream below the weir will be depressed, and the amount of this lowering is denoted by (*e*).

It is now required to find the value of (*e*).

When the water is in a state of uniform permanency, we find as above the following formulas:

$$\frac{\Delta Q}{Q} = \frac{\left(1+\frac{\Delta F}{F}\right)^{3/2}}{\sqrt{1+\frac{\Delta F}{F}}} - 1,$$

and:

$$\frac{\Delta F}{F} = \frac{\Delta a}{a}\left(1+\frac{\Delta a}{b}\operatorname{cotg}.\beta\right)$$

as also:

$$\frac{\Delta p}{p} = \frac{2\Delta a}{p \sin.\beta}.$$

Let Q_0 = the entire amount of water carried down by the stream;
Q = the " " " " flowing over the dam, and
hence: $Q_0 - Q$ = " " " " " carried away by the ferder.
We therefore have also:

$$\frac{\Delta Q}{Q} = \frac{Q - Q_0}{Q_0}$$

But as we consider (Δa) positive when the surface is raised, and negative when lowered, we must also have:

$$\Delta a = -e, \text{ and hence:}$$

$$\frac{Q}{Q_0} = \frac{\left[1 - \frac{e}{a}\left(1 - \frac{e}{b}.\operatorname{cotang}.\beta\right)\right]^{3/2}}{\sqrt{1 - \frac{2e}{p.\sin.\beta}}}.$$

The value of (e) is to be found from this equation by trial, but it may also be determined in the following manner. Assume the banks of the stream below the dam to be so steep that (β) approaches a right angle,
" $\sin.\beta$ " " unity $= 1$, and
" $\operatorname{cotg}.\beta$. " " 0; further assume that the breadth (b) is very great in comparison to the mean depth (a), so that we may place: $b = p$, neglecting $\left(-\frac{e}{b}.\operatorname{cotg}.\beta\right)$ and placing: $p.\sin.\beta = b$ We thus have:

$$\frac{Q}{Q_0}\left(1 - \frac{e}{b}\right) = \left(1 - \frac{e}{a}\right)^{3/2}; \text{ or } \frac{\left(1 - \frac{e}{a}\right)^{3/2}}{\sqrt{1 - \frac{2e}{b}}} = \frac{Q}{Q_0},$$

which may be written, on account of the small value of $\frac{2e}{b}$, like the above expression, which was wrongly placed first; and hence, by developing the 2d member, and neglecting all the terms after the second:

$$\frac{e}{a} = 1 - \left(\frac{Q}{Q_0}\right)^{3/2}\left(1 - \frac{2}{3}\frac{e}{b}\right).$$

If we neglect for the present the term $\left(\frac{2}{3}\frac{e}{b}\right)$, we shall obtain an approximate value for (e), which being placed in the original or general formula, gives:

$$\frac{e}{a} = 1 - \left(\frac{Q}{Q_0}\right)^{2/3}\left\{1 - \frac{2}{3}\frac{a}{b}\left[1 - \left(\frac{Q}{Q}\right)^{3/2}\right]\right\}.$$

This equation may be put under the form: $x + y\frac{a}{b}$, where

$$x = 1 - \left(\frac{Q}{Q_0}\right)^{2/3}, \text{ and } y = \frac{2}{3}\left(\frac{Q}{Q_0}\right)^{2/3}.x$$

A Table giving the values of $\frac{Q}{Q_0}$, x and y, is often of great service. The following was arranged by Prof. Grashof.

Table.

$\frac{Q}{Q_0} =$	0,2	0,3	0,4	0,5	0,6	0,7	0,8
$x =$	0,658	0,552	0,457	0,370	0,289	0,212	0,138
$y =$	0,150	0,165	0,166	0,155	0,137	0,111	0,079

If these values are taken from the Table, and substituted in the formula, the value of (c) will readily be found.

The value of (Q) may also be determined by considering the flow of the water over the dam as the flow from a rectangular orifice; but since it is very difficult to obtain all the necessary elements of the calculation by direct observation; and furthermore, since the coefficients and constants which enter into the determination of the amount of water flowing thro' a rectangular orifice, have always been deduced from experiments on a comparatively small scale, with the water, moreover, flowing over a sharp edge, the values thus found with the aid of these ordinary formulas and co-efficients, are not very reliable.

Let us next consider an Over Fall Dam, and let:

Q_0 = the amount of water carried down by the stream;

Q = " " " flowing over the Dam, and

$Q_0 - Q$ = " " " carried away by the feeder. Also let:

a = the mean depth of the water;

(h) = the height of the dam's crown above the original surface;

c = the lowering of the original surface below the dam, which may be calculated as above;

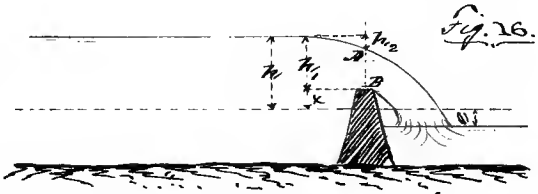
h = the height of the surface after being raised above the original surface, measured so far above the dam that the curvature as it flows over, is not appreciable;

h_1 = the height of this surface, measured at a similar point, above the crown of the dam;

h_2 = the fall of a point vertically over the crown of the dam;

and b_1 = the breadth (or length) of the weir's crown, which may be greater or less than the breadth of the stream above or below the dam.

Fig. 16.



In the problem now to be solved, we have given, Q_0, Q, a, b, b_1, and c, to find (x).

We will consider the flow of the water over the dam as if it passed thro' a rectangular orifice having for its upper edge, the horizontal line thro' A, and for its lower the crown B of the dam. For a little way above the dam, we may regard the velocity of the water as $=0$; but as it nears the weir, its velocity begins to be appreciable, and is considerable already at the point where the over fall curve begins. Let us place the height due to this latter velocity $=(k)$.

From the formula for the flow of water through rectangular

orifice, we have, when the water surface in the reservoir has sunk so far that the curve just touches the upper edge of the opening:

$$Q = \tfrac{2}{3}.\mu.b_1\sqrt{2g}\left[(h_1+k)^{3/2} - (h_2+k)^{3/2}\right],$$

where the ~~ratio~~ of (k) has the same signification as above. This formula, however, can not be applied in its present form as the value of (h_2) is not known.

The coefficient (μ_1) also may be different, and we must therefore make use of an empirical coefficient; which after being combined with the factor $\sqrt{h_2}$ is denoted by (μ_1)

$$Q = \mu_1.b_1\sqrt{2g}\left[(h_1-k)^{3/2} - k^{3/2}\right].$$

After having calculated the value (h_1) from this formula, we will have for the value of (x):

$$x = h_1 - \left[\frac{Q}{\mu_1 b_1\sqrt{2g}} + k^{3/2}\right] + k.$$

In conformity to our above definition of (k), we will now place it: $k = \frac{v_1^2}{2g}$, where (v_1) denotes the velocity at the commencement of the curve. But we also have:

$v_1 = \frac{Q_0}{F_1}$ or $= \frac{Q}{F_1}$, although we may generally place it equal to the first of these values, viz: $\frac{Q_0}{F_1}$, — F_1 denoting the area of the cross-section of the water through the crown of the dam.

When $x > (-e)$, the dam will be a complete overfall.

Let us next consider a partial overfall Dam (Grundwehr).

Here the coefficient (μ_1) depends upon a great variety of circumstances, the form of the dam whether circular, straight or angular in its direction, the nature of the banks, etc., etc.; From various experiments, however, we may with our present knowledge generally place it:

$$\mu_1 = 0{,}35.$$

The calculation of $(x) = -(x)$ is thus performed; let

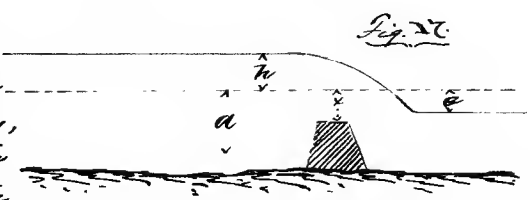
a = the original mean depth of the water;

e = a lowering caused by the feeder, which may be calculated as before; and

x = the depth of the dam's crown below the original water surface.

The cross-section of the water flowing over the weir may be conceived as consisting of two portions, — the upper, which lies above the depressed water surface, flowing freely, and the lower which strikes against the cross-section of the depressed portion of the stream and has to overcome the resistance due to the latter's pressure.

Fig. 27.



The quantity of water flowing through the upper part of this cross-section, may be expressed by:

$$Q_1 = \mu_1 b_1 \cdot \sqrt{2g}\left[(h+e+k)^{3/2} - k^{3/2}\right],$$

and that flowing thro' the lower part, by:

$$Q_2 = \mu b_1 \cdot (x-e)\sqrt{2g(h+e+k)},$$ and hence the total amount by:

$$Q = \mu_1 b_1 \sqrt{2g}\left[(h+e+k)^{3/2} - k^{3/2}\right] + \mu b_1 (x-e)\sqrt{2g(h+e+k)}.$$

In order that the weir shall be only a partial over-fall, we have the condition that: $x > e$.

The co-efficients (μ_1) and (μ) are here again as yet very indefinite and unreliable, on account of the difficulty of making the experiments of this nature upon a sufficiently large scale. In general, however, we may place:

$$\mu = 0{,}8 \quad \text{and} \quad \frac{\mu_1}{\mu} = 2/3.$$

From the above equation for Q, we may now write the value of (x):

$$x = \frac{Q}{\mu \cdot b_1 \sqrt{2g(h+e+k)}} - \frac{\mu_1}{\mu} \cdot \frac{(h+e+k)^{3/2} - k^{3/2}}{\sqrt{h+e+k}}.$$

Before deciding upon the location of a proposed hydraulic motor, the circumstances of the case should carefully be studied; and one of the most important points to be considered is the stream with all its peculiarities which is to supply the motor with water. In estimating the volume of water carried by this stream, it is not absolutely necessary to have a great number of cross-sections, etc.; but at least one cross-section in a normal state of the stream must be taken carefully, and the velocity (c) and the quantity of water (Q) measured.

To find (c) = the mean velocity, the velocity at 1/2 the depth is measured at a number of verticals through the cross-section; these are then combined by various known means into the mean velocity of the entire cross-section, and as the latter is also known, the quantity of water (Q) is then easily found.

We may, moreover, ascertain (Q) experimentally by building a temporary dam of boards, &c., and allowing the water to fall over it. By bevelling off the edge or crown of this dam very sharply, and having vertical sides, the case will be precisely like that in which the co-efficients have been determined by direct experiment. The above formula may then also be used, placing instead of (k), since: $k = \frac{v^2}{2g}$, and $v = \frac{Q_0}{F_1}$, — the value:

$$k = \frac{1}{2g}\left(\frac{Q_0}{F_1}\right)^2$$

M. Castel instituted a series of experiments at Toulouse, France, using, however, a sharp overfall edge and having the ratio $\frac{b_1}{b} = \frac{1}{3}$. He thus found:

$$\mu_1 = 0{,}381 + 0{,}062\frac{b_1}{b}.$$

Weisbach uses another formula, placing:

$$Q = \mu_2 b_1 h_1 \sqrt{2gh_1}.$$

When $b_1 < b$, he finds:

$$\mu_2 = \mu_0(1 + 1{,}718\, n^4)$$

If $b_1 = b$: $\mu_2 = \mu_0(1{,}041 + 0{,}369\, n^2)$,

where (n) denotes the ratio of the cross-section $(b_1 h_1)$ to the entire cross-section F_1 above the dam; that is $n = \frac{b_1 h_1}{F_1}$.
The formula is, however, applicable only when: $n < \frac{1}{2}$.

According to Poncelet and Lesbros we should place for:

$h =$ centimetre	1.	2.	3.	4.	6.	8.	10.	15.	20.
$\mu_0 =$	0,424;	0,417;	0,412;	0,407;	0,401;	0,397;	0,395;	0,393;	0,390

Bornemann also made some experiments with a water way whose breadth: $b = b_1 = 1{,}30$ metre, and obtained:

$$Q = (0{,}567 - 0{,}124\sqrt{n})\, b h_1 \sqrt{2gh_1}$$

where (n) has the same signification as in Weisbach's formula, viz: $n = \frac{b h_1}{F_1}$.

Which of these two formulas for the value of (Q) is the more correct, can only be ascertained by other experiments.

The uncertainty attending the determination of the mean velocity (C) is not of very great moment here, since most streams are subject to all kinds of fluctuations in level. It is, however, necessary to a mean and the extreme values of Q; and hence our observations ought to be repeated at various periods throughout an entire year, the dam of course being built substantial enough to last for this length of time.

Should this involve too great expense, a single cross-section and observations at the dam, when the stream is in a mean or normal condition, will suffice, as the increase or diminution in volume can be calculated from the preceding formulas with the assistance of a few observations and levels when the stream is in an abnormal state.

In all kinds of water works, certain resistances due to an improper disposition of the various parts occur, which may all be reduced to a certain loss of head.
Denote the entire quantity of water by (Q) and the entire fall between two given points in a stream by (H). The theoretical amount of work, or that which we have at our disposal, that may then be performed is:

$$1000.QH = E_0$$ where E_0 denotes

The whole of this effect can not be applied directly at the motor as various obstacles and impediments are encountered, such as the friction of the water in the feeder, etc., etc; and the question then arises as to the length and other peculiarities which this feeder might to possess, so as to make this loss of head due to the resistances, too great velocity, etc., a minimum.
When we have given the required power and have only a small quantity of water at our disposal, the fall H must be made as large as possible. Again, we may have a stream carrying a great volume of water, but having very little fall. This case is not so advantageous for the requirements of a hydraulic motor as where the volume of water is more moderate and the fall greater.
When the required power is given, the necessary head or fall may be computed from Redtenbacher's formula

$$H = 1 + \frac{N}{10},$$ where N denotes the required number of horse powers. This formula is strictly empirical and was deduced from works actually carried out and in operation.
In laying out the works for operating a hydraulic motor, we may follow one of three plans.
(1°): To apply the motor near the dam, thereby having a short feeder and long outlet; or:
(2°): To locate the motor at a distance from the stream and convey and discharge the water by feeder and outlet; or
(3°): To build a dam at some intermediate point and convey the water as before.

In the first case, a great portion of the entire fall (h) must be sacrificed in order to convey the water with the necessary velocity from the upper point (a) to the lower (b). Furthermore it is always preferable to build the factory at some distance from the stream so as to avoid the danger of inundations &c.
In the second case, the loss of head consists of

In the third case the loss of head consists of three parts, the falls h_1 and h_2 of the feeder and outlet, and

In general, other circumstances remaining the same, the third case is that to be preferred in planning water works. Furthermore, the water surface in the feeder is much more uniform and the volume more constant when it is taken from above a dam than when the feeder branches directly off from the stream without the help of a weir of some kind.

Let Q_0 = the original normal quantity of water;
ΔQ_0 = " increase of the " " " " when a new state of uniform permanency has set in; and
a = the mean depth of the normal stream.

Hence, as we have found above:

$$\frac{\Delta Q_0}{Q_0} = \frac{3}{2}\frac{\Delta a}{a} \qquad \text{---------- (1.)}$$

When the water flows over an overfall dam, the quantity so flowing was above found to be:

$$Q = \mu_1 b_1 \sqrt{2g}\cdot h_1^{3/2}.$$

But if the water level varies, this quantity (Q) must also vary; and we therefore have:

$$\Delta Q = \mu_1 b_1 \sqrt{2g}\cdot \tfrac{3}{2} h_1^{1/2}\,\Delta h_1, \text{ and hence:}$$

$$\frac{\Delta Q}{Q} = \frac{3}{2}\cdot\frac{\Delta h_1}{h_1} \qquad \text{---------- (2).}$$

It is necessary however, that the feeder shall receive a constant supply of water ($Q_0 - Q$) in all states of the stream, which is accomplished by means of suitable gates. This supply being constant, we have:

$$\Delta Q_0 = \Delta Q;$$

and if we divide equation (2) by eq. (1), there follows:

$$\frac{Q_0}{Q} = \frac{a}{h_1}\cdot\frac{\Delta h_1}{\Delta a}, \text{ or}$$

$$\frac{\Delta h_1}{\Delta a} = \frac{Q_0}{Q}\cdot\frac{h_1}{a}.$$

This ratio is generally less than (1) unity, and the state of the water in the feeder is therefore more uniform when it is taken from above a dam. In all cases, it is well to make the length (b_1) of the dam's crown as great as possible, — which may be done by running it obliquely across the stream, &c.

We must now endeavor to find suitable mean velocities for the water in the feeder and the outlet

Let (C_1) be the mean velocity of the water in the feeder, and (C_2) " " " " " " " " outlet.

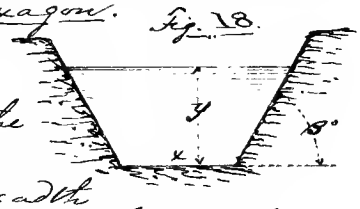
But when these velocities are great, the earth-work of the canale suffers considerably; while if the current is too slow, the dimensions of the feeder must be enlarged to admit of the passage of the required volume, which evidently enhances the cost of the undertaking. Sediment and weeds, moreover, may collect in this latter case, and thus obstruct the free motion of the water.

In general, the velocity of the water in the outlet is made greater than that in the feeder, — that is $C_2 > C_1$, — so that firstly, the water may be removed with sufficient rapidity after having passed thro' the hydraulic motor, and secondly that any rise in the stream's level may not cause an injurious "back-set" in the outlet and thus affect the working capacity of motor. In ordinary circumstances, a portion of the entire fall (H) is used in order that a small water-fall may be formed at the junction of the outlet and the stream.

The feeder should generally be made very short relatively to the outlet, on account of: (1°) the loss due to the filtration of the water thro' the walls of the canal, which, in this portion of the work, are generally embankments and have therefore a greater absorptive power than when the channel is excavated from the solid original earth, as in the case of the outlet, which can therefore also bear a greater velocity of the water without injury to its banks; and (2°) on account of the facility with which the water can be regulated at the bulk-head, thus avoiding the expense of a special keeper, or the loss of time in going continually over a great distance.

Accordingly after having fixed upon the respective velocities (C_1) and (C_2) as well as the respective lengths (l_1) and (l_2) of the feeder and outlet, their cross-sections (F_1) and (F_2) may then be determined from the quantity of water (Q).

We come now to the determination of the proper form of these cross-sections. They are usually made trapezoidal, and in the form of the half of a regular hexagon. Fig. 18.



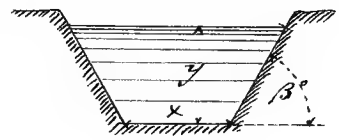
This is the cross-section by which the friction of the water is reduced to a minimum.
Another reason for electing this form is the suitable angle formed by the side slopes
In the ratio $(m) = (\frac{x}{y})$, where (x) is the breadth on the bottom and y the depth of the water. Shows that the

greater we make (m), the broader and hence more costly the channel; while if (m) is made very small, then the water will be inordinately deep and the channel very narrow. Redtenbacher places empirically:

$$m = \frac{x}{y} = 2{,}7 + 0{,}9\,F_1$$

for the feeder; but this ratio may be taken somewhat less for the outlet, as the filtration, &c., of the water is here of no consequence.

After having selected the proper angle (β) of the slopes and determined upon the area of the cross-section (F), we may proceed to the calculation of the dimensions (x) and (y) of the latter. From the figure we see that:



$$F = (x + y.\,\mathrm{cotg.}\beta)\,y = (m + \mathrm{cotg.}\beta)\,y^2,$$

whence:

$$y = \sqrt{\frac{F}{m + \mathrm{cotg}\beta}}, \text{ and } (x) = m\,y$$

We also have the wetted cross-profile:

$$p = x + \frac{2.y}{\sin.\beta} = \left(m + \frac{2}{\sin.\beta}\right) y\,.$$

For the relative fall, we furthermore have:

$$c = k\sqrt{\frac{F}{p}.\alpha},$$

whence: $\alpha = \frac{c^2}{k^2}.\frac{p}{F} = \begin{cases} \alpha_1 \text{ for the feeder, and} \\ \alpha_2 \text{ " " " outlet.} \end{cases}$

Again: $h_1 = l_1.\alpha_1 =$ the portion of the entire fall (h) used in conveying the water to the motor;

and: $h_2 = l_2.\alpha_2 =$ the portion of the entire fall (h) lost in removing the water from the motor.

The sum of h_1 and h_2 subtracted from (h) will then give the useful or disposable fall at the motor.

HYDRAULIC MOTORS.

We will omit the practical considerations of executing these machines, and will pass directly to their theory. In the following, let:

Q denote the amount of water flowing to the motor in one second;

H = the difference in level between the water surface in the feeder and outlet at the machine;

E_0 = the absolute work performed by the descent of the amount of water Q thro' the height H, which is: $E_0 = 1000 . QH$;

N_0 = the absolute effect or work in horse-powers, which is: $N_0 = \frac{E_0}{75}$, since a horse-power corresponds to a mechanical work of 75 kilogramme-metres in one second;

There are, however, a number of sources of loss in work due to: (1°):– the manner in which the water enters the machine; (2°):– the loss of the living force of the water which it still retains as it leaves the machine; – (3°):– a loss of the amount of water in consequence of leakage and splashing; and (4°):– the various kinds of friction, &c., which attend the operation of all machines. The actual amount of work (E) is therefore less than the theoretical or absolute work (E_0). In like manner, the actual number of horse-power (N) is less than the theoretical number (N_0); hence denoting by (η) a coefficient of loss, we may place:

$$E = \eta E_0, \text{ and}$$
$$N = \eta N_0.$$

The work performed by the gravity of the water may be due to either a sudden change in its velocity through impact, etc., or by a continuous motion. The water itself may act in several ways; (1°):– by sinking or descending along with a portion of the machine; (2°):– by hydrostatic pressure; and (3°):– by conversion of the work into living force, which may take place either inside or outside of the machines.

Hydraulic motors may therefore be divided into two classes, – Water-wheels and Pressure engines, and the former again into wheels with horizontal or vertical axes, – i.e., into Vertical Water-Wheels proper, and Turbines.

Water Wheels proper have a horizontal axis and a slow angular velocity; the water moreover enters the machine at a fixed point and leaves it also at such a point.

Turbines have a vertical axis, a great angular velocity, and the water enters and leaves the machine at different points.

Let us first consider the vertical Water wheels

The buckets or paddles are congruent material surfaces at equal distances apart and having the same breadth, which are so disposed on the circumference of the wheel that upon rotation they will generate a circular ring-like body or volume called the crown (Rad-Kranz) or rim.

In this volume, we distinguish the two cylindrical surfaces on the inside and on the outside, and also the two plane rings. The wheel's bed (Rad-boden) is the inner cylindrical surface when conceived as material

Wheels are furthermore distinguished into paddle-wheels and cell-wheels, a cell being the space between two consecutive buckets. The distinction depends upon the form and position of the buckets to the bed of the wheel

The buckets are either plane material surfaces, or broken planes, or cylindrical surfaces.

For water wheels taken in a narrower sense, the buckets or paddles have a general radial direction to the rim, while in the cell-wheels, this direction is more tangential Hence a cell-wheel may be defined as one in which the crown is bounded by material surfaces, and in which the buckets have a tangential direction.

Water-wheels are furthermore divided into wheels freely suspended and those partly enclosed with a circular trough or channel (Kropfräder).

Relatively to the mode in which the water is applied to the wheel, we have:

Wheels with hatches (spannschützen);
-"- -"- overfall sluices, (überfallschützen), and
-"- -"- (Coulissen -"-).

The main distinction in wheels, however, depends upon the point in the circumference where the water strikes or enters, and we thus have:

Over-shot wheels,
Middle-shot or breast wheels, and
Under-shot wheels.

Over-shot wheels are generally freely suspended and provided with common sluices. The water acts directly by its weight, and descends along with the cell which it entered The effect of its impact is here very slight.

Back-shot or high-breast wheels (Rückenschlägige räder) may be either freely suspended, or provided with an envelope or trough (Kropf) and with Coulissen Schützen

The trough need here not have material side walls.

Middle-shot or breast wheels have buckets or paddles, a trough or envelope, and Coulissen Schütze.

If the buckets are bounded by material surfaces at the sides, the envelope must then have side walls.

Under-shot wheels are provided with sluices; the water acts here exclusively by its living force, and impact.

In a Poncelet wheel, this living force acts gradually, and avoids jarrings and shocks as much as possible.

In the following investigations, let:

R = the external radius of the wheel;

a = " " radial dimension or depth of the crown or rim;

b = " " breadth of the wheel at the rim measured parallel to the axis.

When the wheel is enclosed in a trough (gerinne) then let:

a_1 = the radial distance from the wheel's bed or bottom to the inner surface of this trough; and thus:

$a_1 - a$ = the play or space between the outer circumference of the wheel and the surface of the trough. Also let:

b_1 = the breadth of the trough, and hence:

$b_1 - b$ = the space or play between the sides of the crown and the sides of the troughs. Also let:

z = the number of buckets;

e = the distance between two consecutive buckets measured along the external circumference of the wheel, and without taking into consideration the thickness of the buckets. We thus have:

$e = \frac{2\pi R}{z}$ Furthermore let:

i = the entrance arc (eintritts-bogen), which is that arc along [illegible] the water enters

n = the number of revolutions of the wheel per minute;

ω = the angular velocity.

Conceive a point in the wheel at a distance = 1 (unity) from the axis. This point describes in one revolution the path: (2π), and in (n) revolutions, the path: $(2\pi n)$; hence its path in one second is: $= \frac{2\pi n}{60}$. But this is simply the angular velocity of the wheel, and we therefore have:

$$\omega = \frac{2\pi n}{60} = 1{,}0472\, n$$

Again, let:

u = the velocity with which the water enters the wheel estimated at the middle of the arc (i);

v = the velocity of the external circumference;

α = the angle between these two velocities

w = the relative velocity of the outer circumference, which

w equal to the resultant of (u) and $(-v)$;

q = the quantity of water flowing into each cell, or the space between two buckets or paddles; this amount is: $q = R \cdot \frac{c}{v}$.

Consider the space cut by a vertical plane to which the axes is perpendicular. The area of the cross section of the body of water taken up with this cell is then denoted by

F = cross-sectional area of the water in a cell,

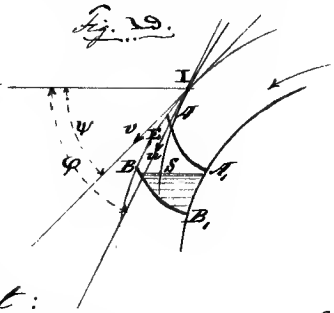
If the buckets are bounded by two side rims then

$F = \frac{q}{c}$ or $= \frac{Q}{b}$.

We must now calculate the power of a wheel and determine the relation by which the coefficient (η) will become as small as possible. For this purpose, we must first consider the losses of work due to the various circumstances attending the wheel's operation, and the disposition of the parts. The first of these is the manner in which the water enters the wheel; and if this is attended by impact, the work so lost is equivalent to a loss of head.

Let Fig. 19, be a section thro' the crown of the wheel; (I) is the middle point of the entrance arc (i). The water as it enters the wheel is in the form of a broad jet or sheet, and we will assume that each separate particle of water has the same relation regarding motion. Let (w) = the absolute velocity with which the water enters. Also let:

Fig. 19.



φ = the angle formed by the direction of (w) with the horizon, and: ψ = " " " " " " " " " $-(v)$ " " " " " "

(v) being the velocity of the outer circumference of the wheel.

As long as the water jet has a free motion, it describes a parabolic path, which is represented by the curve IS.

AA_1 and BB_1 are two consecutive buckets, and their distance apart is denoted by $(c) = AB$. Let E be any point on the outer periphery between A and B. Furthermore, let:

t = the number of seconds elapsed since E passed I;

or assuming that the whole of the water passes thro' the point I, then: t = the time required for the point E with the velocity (v) to advance from I to its present position. If E and I coincide, then at the instant that E passes I, an amount of water enters the cell which is proportional to the entire amount taken up by (q) as BE is to BA. Hence

$q \frac{BE}{BA}$ = the quantity of water taken up in the bucket in (t) seconds.

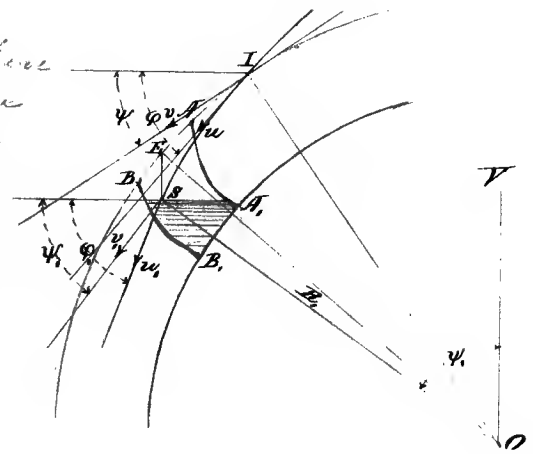
Let us also assume that at the instant when a particle of water strikes the bucket, it becomes in a state of relative rest to the wheel, which is however, not strictly the case.

~~Deriving from this assumption~~ Deducing from this assumption a portion of the entire expenditure of the quantity (q) and denoting this portion by (q'), we have:

$$q' = \frac{q \cdot BE}{B \cdot \pi}.$$

Also let S' denote the point where the last particle given off by the wheel meets the surface of the bucket-water.

The absolute velocity with which the last particle of water impinges is denoted by (v_1), and the angle formed by its direction with the horizon, by (ψ_1).



Now draw a vertical thro' I, and take this line as the axis of X, and the horizontal line SY thro' S as the axis of Y. Denote the height ES of the point E above S by (k); and assume that the:

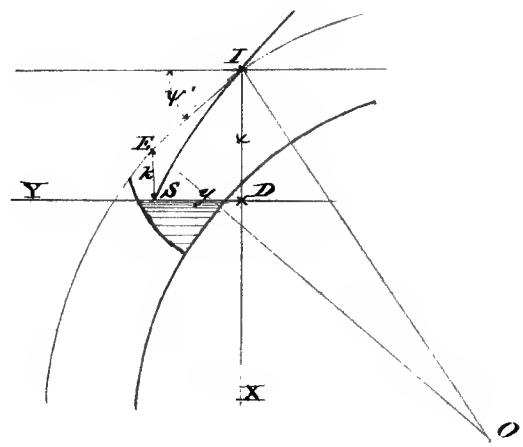
$$\text{arc } EI = \text{chord } EI.$$

Also let (ψ') be the angle formed by this chord with the horizon. With the point I as the origin, let (x) and (y) be the coordinates of the point S; we then have:

$$(1) \text{---} \quad x = vt \cdot \sin.\psi' + k$$

With reference to the angle (ψ'), we have:

$$\psi' = \psi + \tfrac{1}{2} I\hat{O}E = \psi + \frac{vt}{2R}.$$



The distance or height (x) may also be expressed in another manner. For we have:

$$(2) \text{---} \quad x = w \cdot \sin.\varphi \cdot t + \frac{gt^2}{2},$$

where (g) denotes the acceleration of the force of gravity.

If we place these two expressions for (x) equal to each other, and then multiply by $(2g)$, we shall obtain:

$$g^2t^2 + 2gt(w \cdot \sin.\varphi - v \cdot \sin.\psi') = 2gk;$$

whence:

$$gt = -(w \cdot \sin.\varphi - v \cdot \sin.\psi') + \sqrt{(w \cdot \sin.\varphi - v \cdot \sin.\psi')^2 + 2gk}.$$

The positive sign can here only be used to connect the two expressions composing the second member of this equation, since we should otherwise obtain a negative time.

In order to find the time (t) from this formula, we must for the present assume $t = 0$, and calculate an approximate value for (ψ') from the formula: $\psi' = \psi + \frac{vt}{2.R}$;
with this approximate value of (ψ'), an approximate value of (t) may be calculated. This value of (t) is substituted in the formula for (ψ') and a new value of the latter computed; and this process must be repeated until two successive values of (t) are found which are sufficiently equal to each other in value.

After the time (t) is determined, the value of (x) is found from the equation: $x = vt.\sin.\psi' + k$,
in which those values of (ψ') and (k) are to be used, which were last employed in the calculation of (gt). The value of (k) must always be taken from the drawing.

We furthermore have:
$$y = v.\cos.\varphi.t$$
By means of these two values of (x) and (y), the position of the point (S) is fully determined; and if S and O are then connected, the angle (ψ') and the value of (v_1) may also be found.

We may now calculate the loss of head corresponding to the impact of the particles of water during the time (t).

According to "Carnot's proposition", the loss of living force due to an inelastic impact is equal to the sum of the living forces which correspond to the lost velocities; in which we must understand by
the resultant of its velocity before the impact
and the opposite velocity after impact.

For the case in hand, these impacts are repeated during every element of time; but since the element of water is infinitely small in comparison to the wheel, we may assume that the velocity of the wheel remains unaltered, while that of the water particle alone is changed.

The loss of living force is therefore equal to the loss of living force in each water particle; and as the velocity of the particle before impact was (v_1) and after impact (v_1'), it follows that in the sense of Carnot's proposition there is simply the relative velocity with which the water impinges

Denoting this relative velocity of the water particle at the moment of impact by (w_1), its loss of living force is equal to one-half of the product of its mass by $\left(\frac{w_1^2}{2g}\right)$, whence:
$$\frac{w_1^2}{2g} = \text{the loss of living force per 1 Kilogramme of water, which may also be regarded as a loss of head.}$$

But in order to calculate this loss of head which corresponds to the water particle impinging at S, we must first compute (w_1), which is the resultant of $(v_1$ and $-v_1)$; hence:
$$w_1^2 = (v_1.\cos.\varphi_1 - v_1.\cos.\psi_1)^2 + (v_1.\sin.\varphi_1 - v_1.\sin.\psi_1)^2 =$$
$$= (v.\cos.\varphi - v_1.\cos.\psi_1)^2 + (v.\sin.\varphi + gt - v_1.\sin.\psi_1)^2,$$

(I) and (S) are separated from each other by a distance which is very small in comparison to the radius of the wheel (R), so that the velocities (v_1) and (v) of the points (I and S) in the wheel are not essentially different either in magnitude or direction. We may therefore place approximately: $v_1 = v$, and $\psi_1 = \psi$.

In like manner the quotient $\left(\frac{vt}{2R}\right)$ is also so small that we may also place approximately:

$$\psi' = \psi$$

Introducing these assumptions into the formula for (w_1), pag. 37, there follows:

$$w_1^2 = (w.\cos.\varphi - v.\cos.\psi)^2 + (w.\sin.\varphi - v.\sin.\psi)^2 + g^2t^2 + 2gt(w.\sin.\varphi - v.\sin.\psi).$$

But the first two terms are the expressions for the squares of the horizontal and vertical components of the velocity with which the water enters into the buckets, and hence we have:

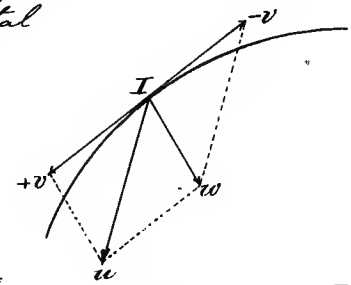
$$w_1^2 = w^2 + 2g.k,$$

and:

$$\frac{w_1^2}{2g} = \frac{w^2}{2g} + k,$$

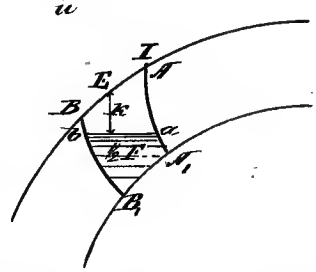
where (k) may also be found by construction.

The value of $\left(\frac{w_1^2}{2g}\right)$ may likewise be found by construction alone. For this purpose, draw a bucket space in such a position that A is just about to pass I; by a horizontal (ab) cut off from this space AA_1B_1B an area $= abB_1A_1 = \frac{1}{2}\,AA_1B_1B = \frac{1}{2}\,I$, and find by a pair of dividers the middle point (E), and the distance (k) of this point above the horizontal line ab.



The value of the mean loss of head under consideration thus found shows under what conditions it will become as small as possible.

This value will be less, the less that of (w); and this relative velocity (w) will diminish the less the angle (α) and hence the nearer the values of (w) and (v) approach each other in direction and magnitude. The value of (w_1) depends also upon that of (k) which varies with (e) the distance between two consecutive buckets, and, other circumstances remaining the same, becomes less the more the buckets are filled. With the same fill, however, the value of (k) will depend considerably upon the position of the point of entrance of the water, and hence also whether the wheel is, <u>over-shot</u>, <u>high-shot</u>, <u>middle-shot</u> or <u>deep-shot</u> wheel.



In an <u>under-shot</u> wheel, the value of (k) may become $= 0$, or even <u>negative</u>. Its value is <u>greatest</u> however in a <u>high-shot</u> or <u>high-breast</u> wheel.

In the foregoing, we have discussed the loss of head due to the impact of the water against the buckets, etc., and assumed that the water was in a state of rest relatively to the wheel immediately after it had impinged. That this latter assumption is not strictly correct may be seen from observation, as an undulatory motion ensues, together with various other complicated whirling motions.

In a Poncelet wheel this swinging motion of the water in the buckets is precisely the object desired, and this wheel will even be made the subject of a separate investigation.

But in addition to the sources of loss of head mentioned above, there are several more. First, relatively to the manner in which the water issues from the feeder or reservoir to the wheel. Denote the coefficient of loss due to this flow from the reservoir by (ζ); and we thus have

$\zeta \cdot \frac{u^2}{2g}$, and if we denote by (h) that portion of the fall (H) at our disposal from the surface of the water in the feeder to the point where the jet enters the wheel, we also have:

$$h = (1 + \zeta)\frac{u^2}{2g}.$$

Here we disregard the velocity (c_1) of the water in the feeder; were this to be taken into account, the height due to this velocity should be added to (h) thus obtaining:

$$h + \frac{c_1^2}{2g} = (1 + \zeta)\frac{u^2}{2g}, \text{ or}$$

$$h = (1 + \zeta -$$

This velocity (c_1) is generally so small in comparison $h(u)$ that we may neglect it, and compensate the error by taking (ζ) a little less.

Second, the loss of head due to the resistance of the air in the cells. Conceive a cell wheel, whose buckets are enclosed at the sides by a rim. As the water enters the wheel, it compresses the air in the buckets which offers thus a certain resistance to the entrance of the water. In cell wheels with a trough channel, this loss of effect is unavoidable; but this may be partly overcome by means of a narrow slit in the bed of the wheel as (S) in the figure. In overshot wheels, this "ventilation" is much more difficult to accomplish, since there is here no trough to exert a counter-pressure, and because as soon as the bucket passes from under the jet, the isolated and compressed air may force a portion of the water out upon its expansion. This may also be partially avoided in these wheels by making the entrance arc (einlauf bogen) i less than the space between two consecutive buckets, and causing the water to enter in a strictly tangential direction to the surface of the bucket.

Let us next consider the losses of work due to the mode in which the water leaves or discharges from the wheel. The living force of the water at the moment it reaches the surface of the outlet is naturally a loss of work to the wheel, and this loss may be considered as consisting of two parts; (1°): the height due to the velocity with which the water leaves the wheel, and (2°): the height of the point of exit above the surface of the outlet.

Let (A) Fig. 29. denote the mean point of discharge; v = the velocity of the circumference of the wheel; w_1 = the velocity of the water as it leaves the wheel and at direction tangential to the casting's surface at the edge, and (w_1) = the relative velocity of the water to the wheel.

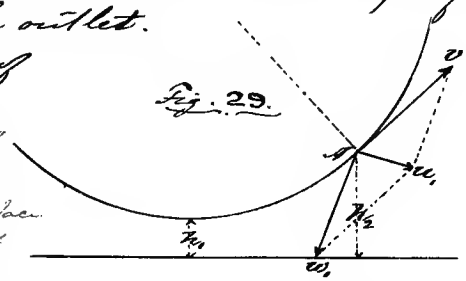


Fig. 29.

In general this relative velocity (w_1) is so small in comparison to (v). that it may be neglected and hence we may put

$$w_1 = v.$$

In a Poncelet Wheel, this relation is different, since here the discharge velocity must be equal to the entrance velocity, and therefore the relative velocity of exit must be much greater.

In a freely suspended wheel, (h_1) is the distance from the surface of the outlet to the lowest point in the wheel's circumference and is generally very small. Let (h_2) = the loss of head due to the untimely discharge of the water from the cells, and take the point (A) as the middle point of the arc along which the water is emptied. In consequence of the centrifugal force, the water surface in the buckets, although supposed to be in a state of rest relatively to the wheel, is a cylindrical surface, which causes this arc to be longer than if the surface were a horizontal plane.

To ascertain that this water surface is cylindrical, let us take a rectangular system of axes, with the origin at the point O, Fig. 30., which we will suppose to be the projection of the wheel's horizontal axis; OXZ is a plane perpendicular to the wheel's axis; the axis OX is horizontal, OZ vertical, and OY perpendicular to the plane of the paper, or in the direction of the wheel's axis. Denote by X, Y, Z, the resultants in the direction of these axes of all the forces acting upon a particle of water whose coordinates (x), (y) and (z) are supposed to be given. Also denote by (r) the distance of this particle from (O), and by (ω) the angular velocity of the wheel, and by (g) the acceleration of gravity.

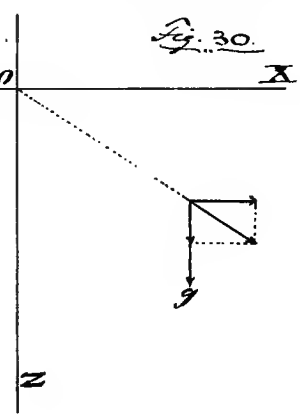


Fig. 30.

Written by ... — "Steam" as an introduction of specific work

$$X.dx + Y.dy + Z.dz = 0.$$

The forces acting upon the particle of water are however:

$$X = \omega^2 x \; ; \quad Y = 0, \text{ and } Z = \omega^2 z + g,$$

and the above equation thus becomes:

$$\omega^2 x\,dx + (\omega^2 z + g)\,dz = 0.$$

Integrating, we have:

$$\omega^2\left(\frac{x^2+z^2}{2}\right) + g.z = Const.,$$

or:

$$x^2 + z^2 + 2.\frac{g}{\omega^2}.z = C_2$$

or:

$$x^2 + \left(z + \frac{g}{\omega^2}\right)^2 = C_3.$$

But this equation is that of a curve in the form of $(x^2 + y^2 = a^2)$ which is the equation of a circle, whose centre lies in the axis Z and at a distance from the origin O which is: $=\left(\frac{g}{\omega^2}\right)$. Repeating this operation for each particle of water, we readily perceive that the surface of the water in the buckets is that of a cylinder, as stated above.

If we denote by R the external radius of the wheel, we then have: $\frac{g}{\omega^2} = g\left(\frac{R}{v}\right)^2$.

In order to calculate exactly the loss of head (h_2) due to the untimely discharge of the water from the buckets, we should be compelled to proceed in the following manner.

A_1B_1, Fig. 32, is the position of the bucket when the water just begins to run out; A_2B_2 the position when the bucket has just been emptied, and AB are intermediate positions. Also let (φ_1) = the angle A_1OZ, (φ_2) = angle A_2OZ, and (φ) = angle BOZ.

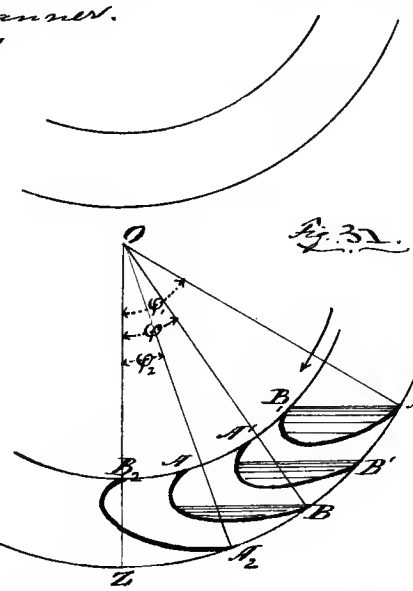


Fig. 32.

Let F_1 = [illegible] the area of the water cross section [illegible] the bucket A_1B_1, and F_2 = [illegible] the area when the water is just [illegible].

The velocity (v) [illegible] be regarded as a function of the angle (φ) and hence we may [illegible]:

$$v = f(\varphi).$$

Now let the wheel turn a very little = $d\varphi$; the amount of water which consequently flows out may be expressed by dV, where V denotes the volume of the water before emptying.

Hence we have:

$$-dV = -f'(\varphi)\,d\varphi$$

This volume of water $(-dV)$ thus falling out is a loss to the wheel, and this loss is equal to the weight of this volume, which is:

$$= \gamma.(-dV).R.(1-\cos.\varphi).$$

Integrating this between the limits (φ_1) and (φ_2), we shall obtain the entire loss of work in one bucket:

$$= \gamma.R\int_{\varphi_1}^{\varphi_2}(-dV)(1-\cos.\varphi) = \gamma R\int_{\varphi_2}^{\varphi_1} dV(1-\cos.\varphi).$$

Now this is the loss of work in one cell or bucket; and in order to find the loss of work per second of time, this must be multiplied by the number of cells emptying themselves in this space of time. But this number is $= \frac{v}{e}$; we therefore have:

$$\gamma.R.\frac{v}{e}\int_{\varphi_2}^{\varphi_1} dV(1-\cos.\varphi) = \text{loss of work per second, which}$$

we may place: $= \gamma.Q.h_2$, whence there follows:

$$h_2 = R.\frac{v}{e}\frac{1}{Q}\int_{\varphi_2}^{\varphi_1} dV(1-\cos.\varphi).$$

But $Q.\frac{v}{e} = q$; hence:

$$h_2 = \frac{R}{q}\int_{\varphi_2}^{\varphi_1} dV(1-\cos.\varphi)$$

$$= \frac{R}{q}\int_{\varphi_2}^{\varphi_1} f(\varphi).(1-\cos.\varphi)\,d\varphi)$$

In this case, we should be obliged to calculate this amount of work last for each element of position of the bucket which is its own.

Let us therefore assume that at the middle point of the arc through which the water is emptied, the water has just half run out of the bucket, and let (φ) denote the angle BOZ. We therefore have:

$$h_2 = R(1-\cos.\varphi)\,\frac{V_1 - V_2}{q},$$

where V_1 denotes the volume of water in the bucket A_1B_1, and V_2 that in the bucket A_2B_2. But $V_1 = q$ and $V_2 = 0$ hence:

$$h_2 = R(1-\cos.\varphi),$$

and (φ) is therefore that value of the angle BOZ which corresponds to the equation: $f(\varphi) = \frac{1}{2}q$; and we accordingly perceive that it is advisable to make this angle as small as possible, since the smaller (φ), the nearer $(1-\cos.\varphi)$ approaches to (0) and hence the smaller the resulting loss of head h_2.

The angle (φ) may also be found by construction in the following manner.

Let AB in Fig 32 be the position of the bucket as the water begins to run out. From the edge A draw a tangent AT and a line AC so that this line will divide the cross section F into two equal portions. The angle CAT will then be the required angle (φ).

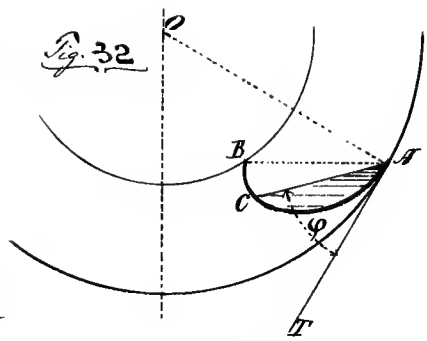


Fig. 32

This construction is on the assumption that the surface of the water in the buckets is a plane.

But if we more properly regard this surface of the water as cylindrical, we must then proceed as follows.

Let as before AB denote the position of the bucket in any arbitrary position. Draw the tangent AT to the circumference of the wheel, and the line AN normal to the line QOZ which passes thro' the centre of the wheel's projection. From the centre O lay off $QO = \frac{g}{\omega^2}$, and thro' A draw AC so that it will cut off equal areas from the circular curve formed by the section thro' the wheel; - that is, the area of the portion above the line should be equal to that below it. The point P is so chosen on this line AC that

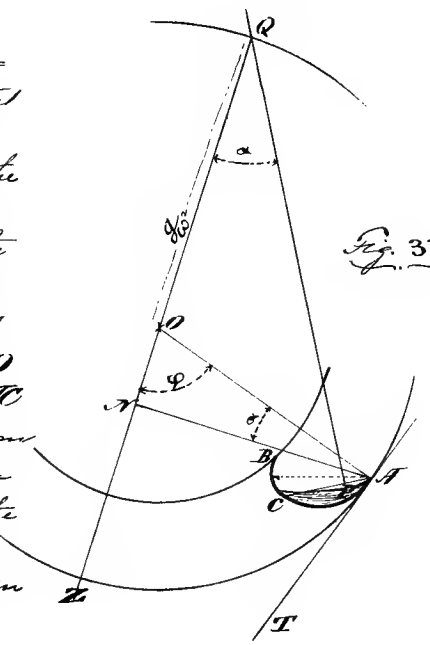


Fig. 33

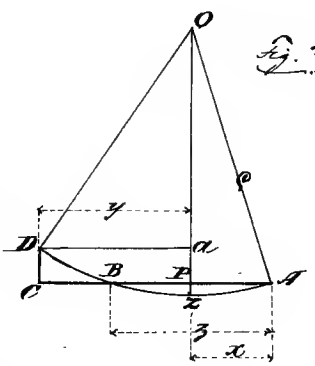


Fig. 34.

For Fig. 34, let AD be the section thro' the cylindrical surface of the water in a bucket and conceive the line AC to be normal to the surface of this bucket, and so drawn that the area DBC is = area BAz. Also let OP, which is drawn perpendicular to AC, be great in comparison to aP. Further let $xP = p$ and $Pa = q$. The area ABz may be conceived to be a segment of a parabolic area on account of the flatness of the arc AD, and hence we may place with not sensible error:

$$\text{area } ABz = \tfrac{2}{3} \cdot 2x \cdot p = \tfrac{4}{3} x \cdot p.$$

We may also place:

$$\text{area } BDC = \square PCDa + \text{area } PBz - \text{area } Daz =$$
$$= y(q-p) + \tfrac{2}{3} x \cdot p - \tfrac{2}{3} y \cdot q = \tfrac{1}{3} yq + (\tfrac{2}{3} x - y) p.$$

Hence we have, from the condition that these two areas shall be equal:

$$\tfrac{4}{3}.xp = \tfrac{1}{3}yq + (\tfrac{2}{3}.x - y)p,$$

whence:

$$(2x - 3y)p = yq$$

and: $p = \frac{x^2}{2\rho}$, and $q = \frac{y^2}{2\rho}$. Substituting these values we shall obtain: $\frac{y}{x} = z$, and

$$z^3 - 2z - 2 = 0,$$ which cubic equation being solved gives:

$$z = 2.$$

In addition to the losses of head mentioned above, there is still another loss – when the wheel has a breast-trough – due to the surface of the water in the lowest bucket being at a higher level than that in the outlet. Denoting this difference in level by (h_1), the work that is performed by the water's sinking thro' this height as it flows out is lost to the wheel. This loss can however not be obviated, as the loss would be much greater if the buckets dipped too deep into the water in the outlet

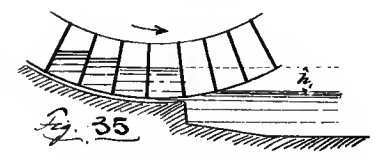


Fig. 35

We next come to those losses of work incurred while the water is passing thro' the wheel. Let (s) be the distance between the outer circumference of the wheel and the interior of the breast trough. In consequence of this play, a certain quantity of water leaks from an upper into the next lower bucket and so on to the lowest. This leakage and the work thereby lost to the wheel, consists in the descent of a portion of water thro' this variable height. Let (z) be the distance from the edge of the bucket to the water's surface in Fig. 36, and (y) the difference in level between the water surfaces in any two consecutive buckets. Further, denote the loss of head due to this loss of work, produced by the passage of a portion of water from one cell into the next, by (h_3); and (φ) the angle formed by the line OS with the vertical; S being the position of any of the slits or rectangular openings between the edge of the bucket and the surface of the breast trough. Also let (v) be the volume of water which passes thro' the slit S from the cell B into the cell A during one second, supposing that the relations of the flow of the water were constant for this space of time.

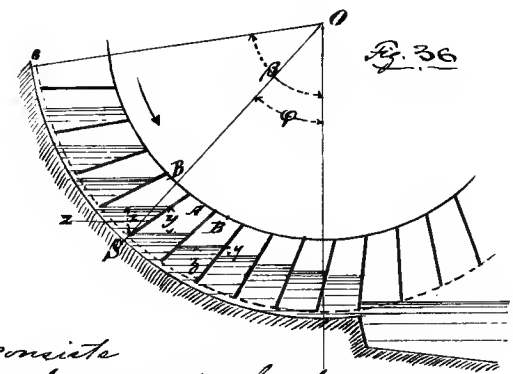


Fig. 36

Let: $V.dt$ = the volume of water actually passing from the cell A into the cell B during an indefinitely small space of time (dt), and let:

$\gamma.V dt$ = the weight of this volume.

This may be conceived as an infinitely thin lamina of water taken from B and transferred into A. In B however, it was in a state of relative rest, and returns into this state in A. The work thus expended is lost to the wheel, while the bubbling and whirling of the water thereby occasioned is converted into heat.

For this work in descending from the water surface in B to that in A we have the expression:

$$= \gamma.V.dt.y. \qquad \text{---------- (1)}$$

For the path described by the slit (S) during the time (dt) we have

$-R.d\varphi$, since $(d\varphi)$ is negative as the angle is diminished.

This path, however, is also equal to the product of the velocity at the circumference by the time; or:

$$-R.d\varphi = v.dt.$$

Deducing from this the value of (dt) and substituting it in eq. (1), we get:

$$\gamma V.dt.y = \gamma.V\left(\frac{-R.d\varphi}{v}\right).y.$$

Next let (β) denote the angle which the radius drawn to the point of entrance of the water makes with the vertical thro' O. The above loss of water and hence of work occure between the limits: $\beta° = 0°$ and $\beta° = \beta°$. The loss of work due to one of these slits (S) would thus be found by integrating the above expression from $\beta°$ to $0°$. The angle (β) however, depends also somewhat upon the form of the buckets.

Integrating the above equation we have:

$$\gamma\frac{R}{v}\int_{\beta}^{0} -V.y.d\varphi = \gamma\frac{R}{v}\int_{0}^{\beta} V.y.d\varphi.$$

This expresses the work lost to the wheel by one of these slits; hence to find the loss during one second, we must multiply the above expression by the number of slits passing along the breast trough during this time. This number is $= \frac{v}{e}$; and hence:

$$\gamma.\frac{R}{e}\int_{0}^{\beta} V.y.d\varphi = \text{loss per second of time} \quad \text{---} \quad \text{---} \quad (2)$$

We must now endeavor to express V as a function of (φ). Let (z) = the head with which the water flows from B into A. For larger values of (φ) we find that $z < y$, and for the smaller values, that $z > y$. This is readily seen by referring to the figure 36.

But if we conceive the water in the cells to be in a state of uniform permanency, the volume of the discharge will be:

$$V = \mu.b.s\sqrt{2gz}.$$

Substituting this value of (V) into equation (2) we obtain:

$$\gamma.\frac{R}{e}\int_0^\beta V.y.d\varphi = \mu\sqrt{2g}\,.\,\gamma\frac{R.b.s}{e}\int_0^\beta y\sqrt{z}.d\varphi.$$

As (y) and (z) are both functions of (φ) they accordingly must remain under the integral sign.
But we have denoted the loss of head due to this source of loss of work by (h_3); hence we must have:

$$\mu\sqrt{2g}.\gamma.\frac{R.b.s}{e}\int_0^\beta y.\sqrt{z}.d\varphi = \gamma.Q.h_3$$

whence:

$$h_3 = \mu\sqrt{2g}.\frac{R.b.s}{Q.e}\int_0^\beta y.\sqrt{z}.d\varphi. \qquad \text{-------} \qquad \text{-------} (3.)$$

The coefficient (μ) may here be placed $= 0{,}8$; as the contraction of the water jet takes place in only one direction, this coefficient ought be taken a little greater than when this contraction occurs in all directions.

To calculate this loss of head (h_3) for a given case, we must know (y) and (z) as functions of (φ). The form of this combined function depends upon the shape of the bucket and its direction against the wheel. Instead of considering the formula for every wheel, it may in general be employed approximately in the following manner.
Divide the angle (φ) into a number of equal parts (n) and let each of these parts $= \Delta\varphi$, and hence: $\varphi = n/\Delta\varphi$.
Place in the formula successively:

$\varphi = \frac{1}{2}\Delta\varphi,\ \frac{3}{2}\Delta\varphi,\ \ldots\ldots\ \beta - \frac{\Delta\varphi}{2}$;
$y = y_1, y_2, y_3 \ldots\ldots\ y_n$;
$z = z_1, z_2, z_3 \ldots\ldots\ z_n$.

We may then calculate the products:

$y_1\sqrt{z_1}, y_2\sqrt{z_2}, y_3\sqrt{z_3}, \ldots\ldots y_n\sqrt{z_n}$,

and hence place:

$$\int_0^\beta y\sqrt{z}.d\varphi = \Delta\varphi\sum_0^n y\sqrt{z},$$

and thus obtain the value of (h_3). From equation (3) we see that this loss of head diminishes directly with (b) and (s).
In general we may place:

$$y = e.\eta, \text{ and } z = e.\zeta,$$

where (η) and (ζ) are coefficients depending upon

$$h_3 = \mu\sqrt{2g}.\frac{R.b.s}{Q}\int_0^\beta \eta\sqrt{\zeta}.d\varphi.$$

It will also be seen that this loss of head (h_3) will be less the smaller the value of the integral. The greater the inclination of the buckets to the radius of the wheel, the less will be the value of (z), and consequently of (ζ), and hence also of (h_3). The value of (η) is not changed.

Under-shot Wheel.

These wheels move in straight troughs, the motion being imparted solely by the living force of the water. The rim on these wheels is usually open, so that the water can escape on all sides.

In estimating the work that this class of wheels can perform, we will simply consider the action of the water only to a depth equal to the depth which the wheel dips into it, as the rest of the water passes under the wheel without producing any result. In addition to the loss of effect from this source, a further loss is produced from a great portion of the water's leaving the wheel with the same velocity as that with which it entered, without having impinged.

To find the quantity of water which passes thro' the wheel without impinging against the buckets, we pursue the following course. Lay off OP and OQ so that:

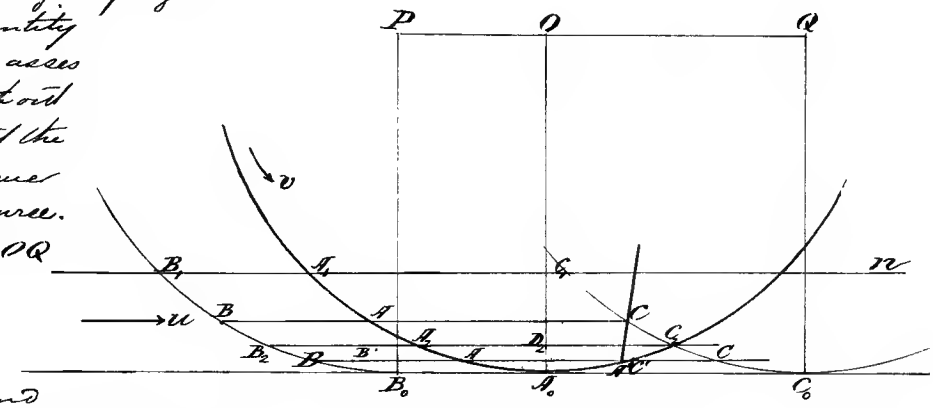
$OP : c = u : v$

or that:

$\frac{OP}{c} = \frac{u}{v}$; (*) and

OQ so that:

$\frac{PQ}{OQ} = \frac{u}{v}$



Let BA be a thread of water flowing to the wheel. When the end A reaches the edge of a bucket, the end B will strike the edge of the following bucket. (2)

Since from the construction: $BA = B_1A_1 = B_0A_0$, the area $A_0B_0B_1A_1$ is the volume of water per unit of breadth which flows to the wheel.

Let us place approximately the line AC = arc AA'. This may may be done without appreciable error, as in reality, the proportions are much greater as far as the wheel is concerned relatively to the depth of the water.

All of the threads above B_2A_2 will impinge with their full length; but one below B_2A, as AB, can only impinge with a part of its length, while the rest flows thro' without coming into contact with a bucket, & hence produces no effect. Suppose that only a length $B'A$ of the thread BA below B_2A_2 impinges; a thread still lower down will impinge with a still smaller length than this, until we get to A_0, where only the particle A_0 strikes while the entire thread passes thro' uselessly. Of the entire volume of water which flows to the wheel, a portion below the thread B_2A_2 will thus pass thro' without impact and is hence a total loss. The area or volume which impinges } is: $A_0A.A_2B_2B'A_0$, which is $= (A_2A_0C_2)\frac{OP}{OQ} =$

$= \frac{2}{3} A_2C_2 . A_0D_2 . \frac{OP}{OQ} = \frac{2}{3} A_0D_2 . OP$, since $OQ = A_2C_2$

the water thread can thus only strike against the surface of the bucket. Denote the point of intersection of AB with the arc described from O by C; that portion of the water thread which will impinge will then be determined by the proportion:

The portion BB' however flows into the bucket space also but does not act, flowing out with the same velocity with which it entered. Hence that volume of water which will impinge against the bucket below the horizontal A_2B_2 is equal to the area $A_0B_2A_2B'$, which is bounded by the geometrical locus (ort) of the points B'. We must therefore find an expression for this area. We have:

$$\frac{AB'}{AC} = \frac{OP}{OQ}$$ (see preceding page).

The points B' form a curve whose terminal points are A_0 and B_2.

$Q = OP.a_1$ = the volume of water per unit of breadth of the wheel which flows into the bucket space in the same time.
Q_1 = the total loss of water " " " " " " .

But $A_0D_2 \cdot OP$ is = to the entire amount of water flowing to the wheel below the thread A_2B_2, and we have thus found that only 2/3 of this amount impinges while the rest is lost, (per unit of length of the wheel's breadth)
(OP) is the path described by a particle of water with the velocity (u) in the same time that a point in the circumference of the wheel describes the path (e) with the velocity (v).
The position of the point (B') may be determined from the proportion:

$$AB' : AC' = OP : OQ$$

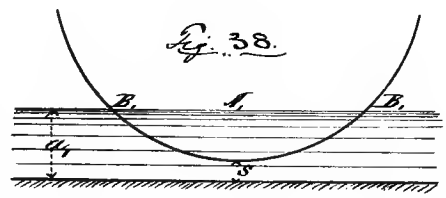


Fig. 38.

Let (s) in Fig. 38, denote the play between the wheel's circumference and the bottom of the trough; B_1A_1 the surface of the water supply, and (a_1) the depth of this water. The volume per unit of breadth of the wheel flowing unused underneath the wheel during the time that the latter describes the path (e) is: (see Fig. (25(c))):

$$A_0B_0 \cdot s = OP \cdot s$$

We thus have for the relative loss of water:

$$\frac{Q_1}{Q} = \frac{\frac{1}{3}OP \cdot A_0D_2 + OP \cdot s}{OP \cdot a_1} = \frac{1}{a_1}\left(s + \frac{1}{3}A_0D_2\right)$$

But since A_0D_2 is very small, we may place:

$$A_0D_2 = \frac{A_2D_2}{2R} = \frac{OQ^2}{8R}, \quad \text{---------- (3).}$$

as $A_2D_2 = \frac{1}{2}A_2C_2 = \frac{1}{2}OQ$.
Now from the proportion:

$$\frac{OP}{e} = \frac{PQ}{OQ} = \frac{u}{v}, \quad \text{---------- (1).}$$

there follows:

$$\frac{OQ}{PQ - QO} = \frac{v}{u - v},$$

and hence:

$$\frac{OQ}{OP} = \frac{v}{u - v}. \quad \text{---------- (2).}$$

Dividing eq. (2) by eq. (1), we obtain:

$$\frac{OQ}{e} = \frac{u}{u - v},$$

which value of (OQ) substituted in eq. (3) causes the latter to become:

$$A_0D_2 = \frac{e^2}{8R}\left(\frac{u}{u - v}\right)^2;$$

the relative loss of water in an undershot wheel is therefore:

$$\frac{Q_1}{Q} = \frac{1}{a_1}\left[s + \frac{e^2}{24R}\left(\frac{u}{u - v}\right)^2\right].$$

From this we perceive that this relative loss will diminish the greater the radius (R), and the less the velocity (v) in comparison to the velocity of the water (u); it also varies directly with the square of (e), which is the distance between two consecutive buckets or paddles

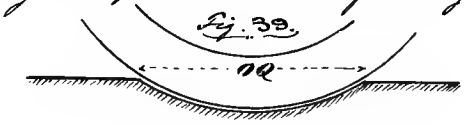
In the above investigation, we have tacitly assumed that: $A_2D_2 < a_1 - s$. If however $A_0D_2 > a_1 - s$, we should then have the relative loss of water:

$$\frac{Q_1}{Q} > \frac{1}{a_1}\left(s + \frac{a_1 - s}{3}\right), \text{ and hence also: } \frac{Q_1}{Q} > \frac{1}{3} + \frac{2}{3}\frac{s}{a_1}$$

In order to diminish the loss of water, the trough is hollowed out cylindrically, as shown in Fig. 39; this hollowing should be large enough to cover the arc $A_2A_0B_2$, as all the particles of water below the line A_2B_2 impinge only in part. Analytically the length, or rather the chord, of this hollow should be:

$$OQ = e\left(\frac{u}{u - v}\right)$$

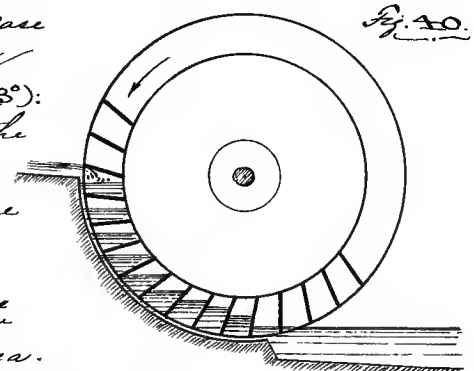
Fig. 39.



Friction of the water against the trough.

There are, however, other losses of effect which cannot be so well calculated as the main losses above discussed. These other losses are (1°):— due to the friction of the water along the surface of the trough, in the case trough wheels; (2°):— the loss of work due to the resistance of the air, and (3°):— finally that due to the friction of the trunnions or axles of the wheel in their bearings. We will discuss these losses in their order.

Fig. 40.



Let (W_1) denote the amount of the friction of the water along the trough's surface, per unit of area. Conceive the water to be in a state of uniform permanency, and let (p) denote the wetted cross-profile, and (l) the length of the surface upon which the friction takes place; hence (lp) is the entire surface exposed to the friction of the water, and $(W_1 lp)$ is the amount of the friction produced by the flow of the water over this surface.

But according to Dubuat's Principle, we have:

$$W_1 . p . l = 1000\, F . l . \alpha,$$

whence: $$W_1 = 1000 \frac{F}{p} . \alpha$$

From the experiments of Darcy and Bazin, however, we find that we should place the mean velocity:

$$v = \sqrt{\frac{\frac{F}{p} . \alpha}{a + b . \frac{p}{F}}}, \text{ and we therefore have:}$$

$$W_1 = 1000\left(a + b . \frac{p}{F}\right) v^2$$

The values of (a) and (b) must here be taken for the case when the surface of the trough is made of planed boards; in this case, as we have already seen:

$$a = 0,000\dot{1}5, \text{ and } b = 0,00000\dot{4}5.$$

We thus obtain finally for the value of W_1:

$$W_1 = \left(0,1\dot{5} + 0,00\dot{4}5 \cdot \frac{r}{t}\right) v^2.$$

The velocity (v) in this formula may be taken as the velocity (v) of the wheel's circumference; and the ratio $\left(\frac{r}{t}\right)$ may be taken as the mean radial dimension of the water in the cells or buckets. We may in general place this mean radial dimension $= 0,0\dot{9}^{m}$, whence: $W_1 = 0,2 \cdot v^2$,

where (v) is the velocity of the wheel's circumference expressed in metres per second of time.

As this is however the friction per unit of area only, the above expression must be multiplied by (l.b) in order to find the total amount of friction; accordingly we have the total amount of friction: $= 0,2 \cdot v^2 \cdot l \cdot b$,

since we may place the wetted cross profile $(r) = b$; and the work thus lost to the wheel:

$$L_w = 0,2 \cdot v^2 l \cdot b \cdot v$$

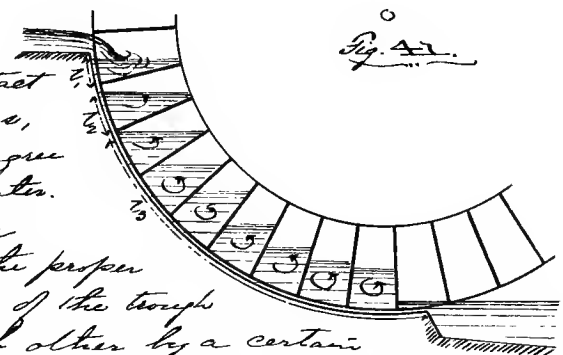


Fig. 41.

The length (l) of the surface of contact consists, however, of a series of arcs, more or less, depending upon the degree to which the cells are filled with water. These arcs ($l_1, l_2, l_3, \ldots$) must be added together in order to obtain the proper length (l). At the entrance portion of the trough these arcs are separated from each other by a certain interval which gradually diminishes, until towards the lower portion of the circumference the arcs form a continuous curve, as seen in Fig. 40 and 41. Practically, these arcs are measured on a drawing in order to obtain the length (l).

But since the water enters the wheel with a certain velocity which is in great measure destroyed by the impact, a circular motion in the direction of the curved arrows in Fig 41 is imparted to the water particles which increases their velocity, relatively to the fixed surface of the trough, over that of the wheel's circumference.

We should therefore take the coefficient in the above expression for the loss of work somewhat greater, and hence we generally place:

$$L_w = 0,\dot{4}.l.b.v^3,$$

(b) being taken as the breadth of the wheel.

Resistance of the Air: — This resistance and hence loss of work to the wheel is greatest, and in fact almost only perceptible, in wheels having an open rim (Rad-Kranz) and radial paddles or buckets. Experiments regarding this resistance were instituted by the French physicists Morin, Piobert, and Didion.

According to the results of their experiments, we must place this resistance, when reduced to the outer circumference:

$$= 0{,}12\, z.a.b.v^2,$$

where (z) denotes -

The loss of work due to this resistance is:

$$E_L = 0{,}12\, z.ab.v^3$$

The resistance of the air consists properly of 3 distinct parts; (1°):- the resistance incurred by imparting to a body of air a certain living force, the work expended in accomplishing this being lost to the wheel; (2°): — the resistance due to the air's being sucked into the cells at the slit for ventilation by the centrifugal force, and then cast out again at the opening or mouth of the cell; and (3°): — the resistance due to the friction of the air upon the circumference and other parts of the wheel.

The total resistance, and hence loss of effect, is greater in open wheels with radial buckets than in cell-wheels, in which it is impossible to be calculated. and hence must be found experimentally This resistance of the air belongs in general to that class of resistance which are not adapted to mathematical or rational investigation, and which are combined into a constant or coefficient which must be determined from observation alone.

Resistance due to the Axle Friction. Let (G) be the weight of the wheel, and (μ) the coefficient of the axle friction. We then have for the amount of this friction, or the resistance:

$$= \mu G.$$

If the two gudgeons or trunnions have different diameters, we may take for the diameter to be brought into the calculation the mean between the two; that is, we place $d = \frac{d_1 + d_2}{2}$; hence we have for the loss of work due to this cause:

$$E_2 = \mu G.\frac{d}{2R}.v$$

The relation between the weight of a wheel and the work that is performed by it, was found by Redtenbacher, from the comparison of a great number of water wheels in full operation, to be:

$$G = 500.N$$

where N denotes the number of horse-powers

For the diameter of the gudgeons or axles, Redtenbacher has also established the following empirical rule:

$$d = 0{,}18\sqrt{\frac{G}{2}} = 0{,}00018\sqrt{2500 . N} = 0{,}0285\sqrt{N}.$$

Placing this value of (d) in the above equation for the loss of work due to the axle friction, and reducing, we obtain:

$$E_3 = 7{,}1 . \mu \frac{v}{R} . N\sqrt{N} = 0{,}6 \frac{v}{R} N\sqrt{N},$$

if we place:

$$\mu = 0{,}085 .$$

When the useful effect or power developed by the wheel is transmitted by means of a cogwheel keyed upon the axle, then the latter is subjected to both flexure and torsion. In this case the heavy iron rack fastened to the rim of the wheel (which forms a considerable portion of the weight of the latter, — when the power is transmitted in that manner) is removed, and hence the weight (G) of the wheel thereby diminished.

The general formula for (E_3) above may however be used for all cases, since when (G) is smaller, (d) will also become less, and the two errors will thus balance each other.

Among those losses of work or resistances which are not susceptible of rational investigation, and which can only be estimated in gross from careful observation, may be enumerated:

(1º): — The spattering of the water, which is thus lost to the cells. The thinner, therefore, the buckets, and the sharper their edge which they present to sheet of water, the less this loss will become. The jet or sheet of water is accordingly also made narrower than the breadth of the wheel.

(2º): — The loss of work due to the resistance and compression of the air in the cells of cell wheels. This may be avoided in part by proper ventilation.

(3º): — The resistance due to the dipping of the buckets into the water in the outlet. This resistance increases with the velocity of the circumference, and the depth to which the buckets dip.

(4º): — The loss of work due to the adhesion of the water to the surfaces of the cells. A certain amount of water remains in the cell after it has been emptied, and is carried upwards, dripping out little by little. The weight thus raised is work which is lost to the wheel. This work also increases with the velocity of the circumference, as then a greater number of particles are raised.

(5º):– The resistance due to the elasticity of the separate elements of the wheel, such as the spokes, &c., and to the imperfections in the construction and erection of the wheel.
In consequence of the weight of the water and the own weight, a certain amount of deformation is produced, although the centre of gravity of the whole is not sensibly changed in position. This deformation gives rise to a general quivering of the wheel, and also to a variable motion.

Some of the above resistances are greater in cell-wheels, others again greater in wheels with open rim and radial paddles or buckets. In wooden wheels, these resistances, and hence losses of effect are in general greater than in the case of iron wheels, for the following reasons:
(1º):– The splashing and spattering of the water due to the greater necessary thickness and clumsiness of the buckets.
(2º):– The greater adhesion of the water to the wooden surfaces.
(3º):– The greater liability to deformation, &c.,.

All of the above mentioned resistances which can not be calculated mathematically, may be comprehended in the general coefficient (ϑ), for which we may place:

$\vartheta = 0{,}02$ for iron wheels, and
$\vartheta = 0{,}04$ " wooden "

[illegible]

Q_1 = the loss of water per second of time;
H_1 = " " " " " head due to
E_1 = " " " " work which is not due to (Q_1) and (H_1).

We may then place:

$$E = 1000(Q - Q_1)(H - H_1) - E_1 .$$

But we have for the theoretical amount of work.

$$E_0 = 1000 . QH ,$$

and for the co-efficient of loss (η) we therefore have:

$$\eta = \left(1 - \frac{Q_1}{Q}\right)\left(1 - \frac{H_1}{H}\right) - \frac{E_1}{E_0} .$$

The loss of head (H_1) consists as we have seen, of 6 parts:

$$H_1 = \zeta \cdot \frac{u^2}{2g} + \frac{w_1^2}{2g} + \frac{v^2}{2g} + h_1 + h_2 + h_3 . \quad \text{-----} \quad \text{----(1)}.$$

The first of these is the loss of head due to the manner in which the water enters the wheel. For a sliding or an overfall sluice (Spann oder überfall schütze), we may place the co-efficient (ζ)

$\zeta = 0{,}1$, and for a "Coulissen" sluice: $\zeta = 0{,}20$ to $0{,}25$

The second of these losses of head, is due to impact or percussion of the water against the wheel, and was found to be: $\frac{w_1^2}{2g} = \frac{w^2}{2g} + k$, where (w) denotes the relative velocity of the water at the middle point (i) of the entrance arc (einlauf bogen) and is the resultant of (u) and $(-v)$; (k) was found by construction, see page 36; and is the vertical distance of the middle point of arc $AB = c$ = to the distance between two buckets, above the horizontal line which divides the cross section of the volume of water in the cell into two equal parts.

The third loss of head is due to

The fourth loss, (h_1) is that which is due to the amount or height of suspension, in freely suspended wheels, or the distance between the lowest point in the wheel and the surface of the water in the outlet.

The fifth loss (h_2) of head is that resulting from the premature discharge of the water from the cells, and is calculated from the formula: $h_2 = R(1 - \cos.\varphi)$,
where (φ) denotes the angle formed by the vertical radius with that which is drawn to the edge of the bucket containing only $(\frac{1}{2}q)$. This angle may also be found by construction.

The sixth loss (h_3) refers only to trough wheels and to the discharge of water from one cell into the next lower one through the space or play between the edge of the buckets and the trough. This loss may be computed from the formula:

$$h_3 = \mu\sqrt{2g}.\frac{R.b.s}{Q.c_o}\int_0^{\beta} y\sqrt{z}.d\varphi.$$

The relative loss of water $(\frac{Q_1}{Q})$ refers only to <u>under shot wheels</u>, and is expressed by:

$$\frac{Q_1}{Q} = \frac{e^2}{24.R\left(\frac{u}{u-v}\right)^2}\Big]\cdot\frac{1}{\alpha_1}$$

When the trough is hollowed out as indicated on page 50, then the value of $(\frac{Q_1}{Q})$ becomes considerably less than the above expression.

Relatively to (E_1), we may place this:

$$E_1 = E_w + E_2 + E_3 + \vartheta.E_0$$

In this expression we have:

$E_w = 0{,}2.\gamma.b.v^3$, for trough wheels only,
$E_2 = 0{,}12.\gamma.a.b.v^3$, = resistance of the air, and
$E_3 = \mu.G.\frac{d}{2R}.v$, = axle friction, which becomes:
$= 0{,}6.\frac{v}{R}.\sqrt{V}$, when (G) and (d) are not known.

For the coefficient (ϑ) we may place:
$\vartheta = 0{,}02$ to $0{,}03$ for iron wheels, and: $\vartheta = 0{,}04$ to $0{,}05$ for wooden ones.

Maximum value of (η): — It may occur that the maximum value of the co-efficient of effect (Wirkungsgrad) (η) is required, when we have given the other elements necessary to fully determine the wheel. This, however, would lead to all kinds of almost insurmountable difficulties; and we therefore prefer to use for the majority of the elements of the wheel, such dimensions and quantities as have borne the test of experience. Such empirical values must however not be relied upon too exclusively, and we should accordingly examine our formulas to ascertain which of these values and elements affect most the value of (η).

With respect to the choice of the kind of wheel, when the necessary data are given, the graphic diagram in "Redtenbachers Resultaten" will prove of great service

We are generally called upon to construct a wheel which shall produce a given number of horse-powers for a given fall

We have here:

$$N = \frac{E}{75} = \frac{\eta E_0}{75} = \frac{\eta}{75} \cdot 1000 . Q H ,$$

whence:

$$Q = \frac{0{,}075}{\eta} \cdot \frac{N}{H}$$

Comparing the results thus found with the diagram, the kind of wheel is determined, and also a few other elements, such as the degree of filling of the buckets, which is:

$$\varepsilon = \frac{Q}{a b . (1 - \frac{a}{2R}) v} ,$$

for which however we may generally place:

$$\varepsilon = \frac{Q}{a b . v} .$$

We have further:

R = the radius of the wheel;
V = " velocity of the water at the moment it reaches the wheel;
e = " space between two buckets;
Z = " number of the "

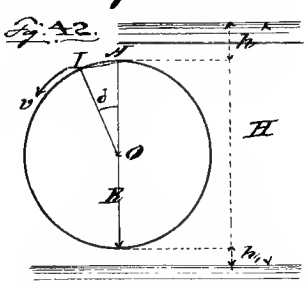
These elements are governed in part by the following fundamental principles.

The radius (R) should not be too great as otherwise the wheel becomes too costly, besides increasing the liability to deformation, &c.

For an overshot wheel, (R) is determined in the following manner. The water is not directed at its entrance to the vertex I of the wheel, but the middle point of the entrance arch is placed at a distance to the left of the vertical diameter at a point (I), the angle $IOA = \delta$ being generally taken from 8° to 12° Also denote by:

h_1 = the distance of the surface of the water in the feeder above this point I;

Fig. A 2.



and by (h_0) :- the amount of suspension. We then have the total head or fall : $H = h + h_1 + B(1 + \cos.\delta)$,
where : $h = (1+\zeta)\frac{w^2}{2g}$, the value of the velocity of the water (w) being taken from $1\frac{3}{4}.v$ to $2.v$.
From the above equation the value of (R) may now be determined.

For a high-breast-wheel, we may place : $R = \frac{2}{3}H$
" " Breast wheel, " " " " $R = H$
$R =$ (from 1,25 to 1,5) H,
$R =$ (" 1,5 " 2,5) H.

In under-shot wheel, there exists no relation whatever between (R) and (H). For a Poncelet wheel, we usually place : $R = 2H$

Periphery Velocity. In under-shot wheels, the velocity (v) can only be calculated from the entrance velocity (w) of the water
In the other classes of wheels, the relation of (v) to (w) may be assumed more arbitrarily. A small velocity should generally be chosen for (v); but if it is taken too small, then the depth (a) and breadth (b) of the wheel must be correspondingly greater, and the cost therefore also increased; and moreover as this slow velocity must generally be considerably multiplied in order to drive the machinery with the required speed, a too slow velocity (v) will furthermore increase the cost of the transmission of power, besides causing too much loss in effect. These two considerations should be made to balance as nearly as possible
For freely suspended cell-wheels, the loss of head H, is determined in great part by the height (h_2) This latter, however, increases with the curvature of the water surface in the buckets, which in turn increases with the velocity (v) For this class of wheels, we may generally take : $v = 1,5$ metres per second.
In trough-wheels, on the other hand, the greater we take the velocity (v) the smaller the breadth (b) of the wheel, and the smaller also the loss of head (h_3). For these wheels, we may take in general :
$v = 1,5^m$ to $1,8^m$ per second

Filling Ratio. This depends upon the following considerations.
The smaller the filling ratio (ε), the greater will be the working coefficient (η); but the greater on the other hand we take (ε), the smaller may we take the dimensions (a) and (b) and hence the less the cost of the wheel, and also the loss of head (h_2).
For freely suspended cell wheels experience has taught that we should generally take : $(\varepsilon) = \frac{1}{12}$ to $\frac{1}{15}$.
In trough wheels, again, the greater we take (ε), the less we may make (b), and hence also the less the loss of head (h_3); but when (ε) increases, the number of buckets (z) also increases However, as

in the expression for [illegible] the factor (z) is under the radical sign, it follows that [illegible] depends more upon the dimension (b) than upon (z).

A limit is however set for the filling ratio (ε) on account of the necessary ventilation of the wheel; and we may therefore take (ε) no greater than. $(\varepsilon) = 1/3$ for high breast wheels, and:

$\varepsilon = 1/2$ " all other wheels.

Dimensions (a) and (b). — For the breadth (b) and depth (a) of the wheel, or more properly of the wheel's rim, we have the condition:

$$a.b = \frac{Q}{\varepsilon.v}$$

For solving this equation, which is the product of two unknown quantities, another equation involving one or both of the latter would ordinarily be required. Redtenbacher has however established the empirical formulas, deduced from the observation of a great number of wheels:

$$\frac{b}{a} = 2{,}25\sqrt[3]{N_0} = 5{,}34\sqrt[3]{Q.H},$$

for cell wheels; and:

$$\frac{b}{a} = 1{,}75\sqrt[3]{N_0} = 4{,}15\sqrt[3]{Q.H},$$

for paddle wheels (schaufel-räder), where in both formulas:

$$N_0 = \frac{1000}{75}.Q.H$$

Division of the circumference. — In general a small space (e) between two consecutive buckets is best, and this leads to a greater number of buckets or paddles (z). For freely suspended cell-wheels opportunity must however be given for the air contained in the cells to escape upon the entrance of the water, and hence the space (e) should be somewhat greater than the entrance arc of the water (i). Here again the empirical formulas or rules established by Redtenbacher prove of great service. He places:

$$e = 0{,}2 + 0{,}7.a \text{ metres};$$

but this must of course be a round number. Also for (z) he has:

$$z = \frac{2\pi R}{e},$$

which number of buckets (z) should however be exactly divisible by the number of arms or spokes in the wheel.

Grashof's Formulas. The following formulas were established by Prof. Grashof, in which the working coefficient (η) is considered as a function of (N, H, and v.) They are also contained in "Redtenbacher's Resultaten" page 172.

For an overshot wheel:

$$\eta = 0{,}91 - \frac{0{,}114.v^2 + 0{,}5\,\frac{\sqrt[3]{N}}{\sqrt{v.H}} + 0{,}1 + 0{,}013.v\sqrt{N}}{H};$$

where $\frac{u}{v} = 2$.

For a high-breast-trough-cell wheel with collision entrance:

$$\eta = 0{,}94 - \frac{0{,}156.v^2 + 0{,}634\,\frac{\sqrt[3]{N}}{\sqrt{v.H}} + 0{,}0084.v\sqrt{N}}{H},$$

where $\frac{u}{v} = 2$.

For a breast wheel with trough and Poncelet entrance:

$$\eta = 0{,}86 - 0{,}0053 v^2 - \frac{0{,}136 v^2 + 0{,}092 \sqrt[3]{N}\sqrt{H} + 0{,}0056 . v\sqrt[3]{N}}{H},$$

when $\frac{u}{v} = 1{,}75$.

For a low-shot trough wheel with an over-fall sluice:

$$\eta = 0{,}87 - 0{,}006 . v^2 - \frac{0{,}130 v^2 + 0{,}052 + 0{,}004 . v\sqrt[3]{N}}{H},$$

when $\frac{u}{v} = 1{,}75$.

For a bow-shot trough wheel with sliding sluice (span schütze):

$$\eta = 0{,}97 - \frac{(0{,}09 + 0{,}008 . H) v^2 + 0{,}002 . v\sqrt[3]{N}}{H} - \frac{0{,}034 + 0{,}05 H}{v} - (0{,}011 - 0{,}002 . H) v^2,$$

where $\frac{u}{v} = 1{,}75$

Example. Let it be required to construct an over shot wheel of 27 horse powers, and with a fall of 12,6 m. We thus have here:

$$N = 27, \text{ and } H = 12{,}6 \text{ m}$$

Choosing the velocity: $v = 1{,}5$ m, we find from the above formula, the working coefficient: $\eta = 0{,}85$,

and for the quantity of water required: $Q = \frac{0{,}075 . 27}{0{,}85 . 12.6} = 0{,}189 = 0{,}19$ cubic m

Consulting Redtenbacher's diagram, we find that these quantities still lie in the sphere of overshot wheels, which is accordingly the proper kind that should be selected under the given circumstances.

After the following dimensions have been found or determined from their respective formulae,

$R = 6{,}0$ m; $a = 0{,}27$ m; $b = 1{,}9$ m; $c = 0{,}39$ m, and $n = 3$ m, we then have: $\varepsilon = \frac{Q}{ab . v} = \frac{1}{4{,}05}$, $= 1/4$,

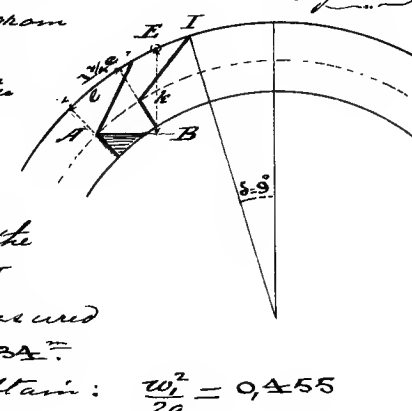
and:

$$\zeta . \frac{u^2}{2g} = 0{,}1 . \frac{u^2}{2g} = 0{,}046 .$$

Having now all the necessary terms or quantities, the value of the working coefficient (η) may be determined more accurately from the former formula for this value on page 54.

To find the value of (K), we draw our diagram to a suitable scale, and cut off by a horizontal line AB one half of the cross section of the water contained in the cell, or what is the same thing one-eighth (1/8) of the whole cross section of the cell ABI, and then let fall upon AB a perpendicular from the middle point E of the space or division immediately to the left of I This perpendicular EB is (K) and when measured by the scale of the drawing, we find: $K = 0{,}34$ m.

Fig. 43



We also have: $\frac{w^2}{2g} = \frac{v^2}{2g} = 0{,}115$, from we obtain: $\frac{w_1^2}{2g} = 0{,}455$

Deducting from the entire head or fall H, the heights: (h) and $R(1+\cos\delta)$ we the amount of suspension:

$$H - h - R(1+\cos.\delta) = h_1 ,$$ the value of (h) here being:

$$h = (+\zeta)\frac{u^2}{2g} = 0{,}505 .$$

Substituting this and the other values found above in the last equation we have: $h_1 = 0{,}169$ m.

If we consider the water surface in the cells as horizontal, we shall then have: $h_2 = R(1-\cos.\varphi) = 0{,}426$ m;

and as (h_3) refers only to trough wheels, it here falls out of the formulas, since we have no trough for an Overshot wheel; hence there follows next from the equation on page 54:

$$H_1 = 0{,}046 + 0{,}455 + 0{,}115 + 0{,}169 + 0{,}426 = 1{,}211 \text{ m},$$

and the relative loss of head:

$$\frac{H_1}{H} = 0{,}096 .$$

We also have:

$E_3 = 21{,}2$ Kilogramme-metres per second, and:

$E_0 = 1000 . QH = 2394$ Kilogramme-metres per second;

and hence: $\frac{E_3}{E_0} = 0{,}009$, or not quite one per cent.

Assuming:

$$\vartheta = 0{,}04 \text{ (per cent.)}$$ we have.

$$\frac{E_1}{E_0} = 0{,}049 ,$$

and therefore:

$$\eta = 1 - 0{,}096 - 0{,}049 = 0{,}855$$

In this case we find that this final and more accurate value of the working co-efficient (η) is so near to the approximate value found in the beginning from Prof. Grashof's formula, that all the dimensions and elements of the wheel found above, and which are dependent in a measure upon this co-efficient, may be retained. Were the last value for (η) considerably divergent from the first approximate value, the above values would have to be calculated over again and corrected with the last value of (η).

Under-shot Percussion Wheels.

In these wheels the buckets or paddles are usually plane surfaces, somewhat inclined to the radius at that point where they are situated, so that a certain amount of the living-force of the water may be neutralized while the rest acts by pressure upon the surface of the paddle. This inclination however must not exceed certain limits in order that the water may be emptied before the edge of the bucket leaves the surface of the water in the outlet, as otherwise there would result a loss of effect due to the height from which the water falls into the outlet. Furthermore, to reduce the ratio $(\frac{Q_1}{Q})$ of the relative loss of water, the wheel is provided with a hollow-cast trough.

Fig. 44.

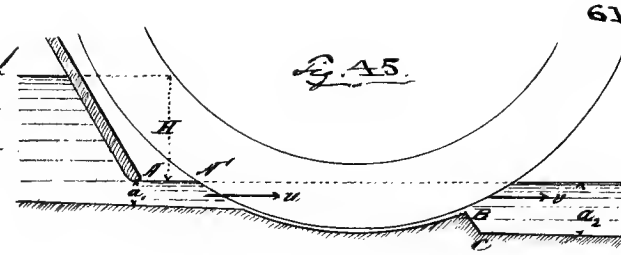


Fig. 45.

If the bottom of the outlet in such a wheel were made on the same level as the bottom of the under conduit, the water behind the wheel would stand higher than the surface of the water in front of it, on account of its slower velocity (v). The bottom of the outlet is therefore made so that the surface of the water in the outlet will stand at the same level as the point A'. This is effected by means of the offset BC in the figure.

The entrance velocity (u) is:

$$u = \sqrt{\frac{2gH}{1+\zeta}},$$

where (ζ) denotes a [illegible]

In order to calculate the useful effect which such an under-shot percussion wheel is capable of performing, we will assume that the paddles are truly radial in their direction.

We may here place:

$$\eta = (1-q)\left(1-\frac{H_1}{H}\right) - \vartheta,$$

in the constant (ϑ) being included the axles friction, the resistance of the air, and those other resistances which can only be estimated in gross. This term (ϑ) may be taken from:

$$\vartheta = 0{,}08 \text{ to } 0{,}10,$$

varying with the velocity and the height H.

For the relative loss of water, we may take:

$$\frac{Q_1}{Q} = 0{,}10$$

For the relative loss of head (H_1), we have:

$$H_1 = \zeta\cdot\frac{w^2}{2g} + \frac{(u-v)^2}{2g} + \frac{v^2}{2g},$$

since we may here take:

$\frac{w_1^2}{2g} = \frac{w^2}{2g}$, where $w = u - v$. The other losses of head, h_1, h_2 and h_3 have here no significance. The last equation becomes:

$$H_1 = (1+\zeta)\frac{w^2}{2g} - \frac{uv}{g} + \frac{v^2}{g} = H - \frac{(u-v)v}{g}$$

We thus have:

$$\eta = (1-q)\frac{(u-v)v}{gH} - \vartheta.$$

This value of (η) becomes a maximum when $(u-v)v$ is a maximum, [illegible]

which $u - 2v = 0$

or $v = \frac{u}{2}$

The most advantageous velocity for the wheel's circumference is however a little less than this, as some of the resistances included in the term (ϑ) increase with (v). Experience has taught that we should take:

$$v = 0{,}4\sqrt{2gH},$$ from which there follows the ratio:

$$\frac{v}{u} = 0{,}4\sqrt{1+\zeta} = 0{,}4\sqrt{1{,}1} = 0{,}42,$$

or a little less than one half ($\frac{1}{2}$).

We accordingly have for the value of the working coefficient (η):

$$\eta = (1-q)\left(\frac{u}{v}-1\right)\frac{v^2}{gH} - \vartheta = (1-q)\left(\frac{1}{0,42}-1\right).0,32 - \vartheta =$$
$$= 0,44(1-q) - \vartheta$$

If we take as an average value: $q = 0,1$, we have:

$$\eta = 0,4 - \vartheta,$$

and taking also as an average value: $\vartheta = 0,08$, we finally obtain: $\eta = 0,32$,

or about (1/3) of the theoretical effect is all that such a wheel can perform.

The useful effect of these under-shot percussion wheels may however be slightly increased by giving to the paddles or buckets a slight inclination to the radius.

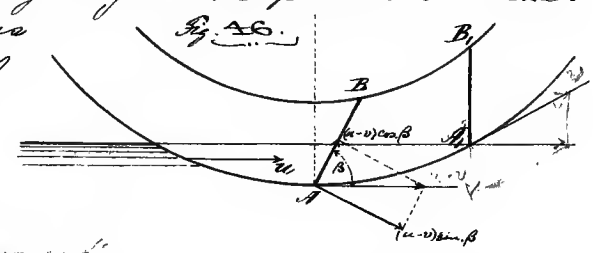
Let us consider the lowest point A of the wheel, and let the paddle be inclined under an angle (β) to the horizon.

The relative velocity $(u-v)$ may be resolved into two components:

$(u-v)\cos\beta$ in the direction of the bucket, and

$(u-v)\sin\beta$ perpendicular to the bucket.

Fig. 46.



We can have here for the loss of head (H_1):

$$H_1 = \zeta.\frac{u^2}{2g} + \frac{(u-v)^2}{2g}.\sin^2\beta + \frac{v^2}{2g}.$$

Multiplying the first and third terms of the second member of this equation by: $1 = \sin^2\beta + \cos^2\beta$, we obtain:

$$H_1 = \left(H - \frac{(u-v)v}{g}\right)\sin^2\beta + \left(\zeta.\frac{u^2}{2g} + \frac{v^2}{2g}\right)\cos^2\beta,$$

whence there follows:

$$\frac{H_1}{H} = \left(1 - \frac{(u-v)v}{gH}\right)\sin^2\beta + \left(\frac{\zeta}{1+\zeta} + \frac{v^2}{2gH}\right)\cos^2\beta,$$

and

$$1 - \frac{H}{H} = \underbrace{\left(\sin^2\beta + \cos^2\beta\right)}_{=1} - \frac{H_1}{H} = \frac{(u-v)v}{gH}.\sin^2\beta + \left(\frac{1}{1+\zeta} - \frac{v^2}{2gH}\right)\cos^2\beta$$

We may here take the same value for the ratio $\left(\frac{v}{u}\right)$ as above, viz: $\frac{v}{u} = 0,42$, whence $\left(\frac{u}{v}-1\right)\frac{v^2}{gH} = 0,44$; and therefore:

$$1 - \frac{H_1}{H} = 0,44.\sin^2\beta + \left(\frac{1}{1,1} - 0,16\right)\cos^2\beta = 0,44.\sin^2\beta + 0,75\cos^2\beta,$$

whence:

$$\eta = (1-q)\left(0,44.\sin^2\beta + 0,75.\cos^2\beta\right) - \vartheta$$

The angle (β) must not be taken greater than 60°. This value gives:

$$\eta = 0,52(1-q) - \vartheta = 0,47 - \vartheta,$$

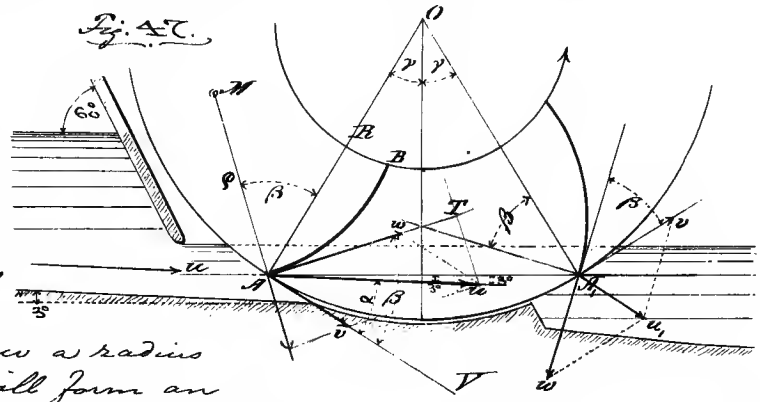
when we take $(q) = 0,1$.

By such an inclination of the buckets, we thus perceive that the useful effect of such a wheel can be increased by at most (7) per cent.

Poncelet's Wheel.

In this wheel, the water enters without percussion along properly curved buckets, and acts solely by pressure. The water is conducted to the wheel in the same manner as to a well ordered under-shot wheel

Fig. 47.



Let A be the center of the entrance arc of the water, and let us now trace out the course of a particle of water at the point A at the moment that the edge of the bucket AB passes this point. From the centre of the wheel (O) draw a radius to the point A, which will form an angle (γ) with the vertical radius. Also draw the bottom of the conduit so that it shall make an angle = 3° with the horizon. To draw the bucket AB, draw the line AM making the angle (β) with AO, and with a radius (ρ) describe the circular bucket AB. Let AT be the tangent to this arc at the point A, and AV the tangent to the circumference of the wheel at the point A. Denote the angle (VAw) formed by the direction of the initial velocity (w) of the water with this latter tangent AV by (α). If the form of the bucket AB were not circular, then M denotes the centre of curvature for the elementary arc at the edge, and (ρ) the corresponding radius of curvature. Also let (R) be the radius of the wheel. From the construction of the figure, since the velocity (u) is parallel to the bottom of the conduit, we then have:

$$\gamma = \alpha + 3°.$$

In order that the particle of water may flow into a vessel, the relative velocity must be normal to the surface of the bucket must become -

The component in the direction of (w) is: — $w.\sin.(\beta-\alpha)$; and that in the direction of (v) is: — — — — — $v.\sin.\beta$

If the resultant of these two velocities shall be equal to 0, we must then have:

$$w.\sin.(\beta-\alpha) = v.\sin.\beta,$$

and the water will then enter the bucket without percussion.

The component of the initial velocity of the water (w) in the direction of the tangent AT is:

$$w = w.\cos.(\beta-\alpha) - v.\cos.\beta$$

The water therefore enters the wheel with the velocity (w), and ascends the surface of the bucket until by the combined action of gravity and the centrifugal force, this velocity becomes = 0; the water particle then begins its descent and leaves the wheel at the same height A, as

it entered; and for the present, we will also assume that its exit velocity is the same as its entrance velocity (u).

The useful effect or work which a Poncelet wheel is capable of performing is:

$$E = 1000(Q - Q_1)\frac{u^2 - u_1^2}{2g} - E_1 \quad \text{per second} \quad \text{---------- (1).}$$

The term (E_1) may here be regarded as empirical method of being made up of the various disturbances to which can not be made the subject of calculation.

The absolute or theoretical effect is:

$$E_0 = 1000.QH \quad \text{---------- (2).}$$

and we therefore have by dividing equation (1) by eq. (2) the working coefficient:

$$\eta = (1-q)\frac{u^2 - u_1^2}{2gH} - \nu, \quad \text{where we place } \frac{E_1}{E_0} = \nu$$

From the parallelogram of forces in Fig. 41, we also have:

$$u^2 = v^2 + w^2 + 2v.w.\cos.\beta,$$

and: $u_1^2 = v^2 + w^2 - 2v.w.\cos.\beta$, whence by subtraction there follows: $u^2 - u_1^2 = 4.v.w.\cos.\beta$.

But because

$$v + v.\cos.\beta = u.\cos.\alpha - v, \text{ and hence}$$

$$u^2 - u_1^2 = 4.v(u.\cos.\alpha - v).$$

We thus have:

$$E = 1000.Q(1-q)\frac{2.v(u.\cos.\alpha - v)}{g} - E_1,$$

and:

$$\eta = (1-q)\frac{2v(u.\cos.\alpha - v)}{g.H} - \nu. \quad \text{---------- (3)}$$

The term (ν) is best to be found empirically. In general we may place approximately: $\nu = \frac{10}{100} = 0{,}1$

Maximum value of (η). — Let it next be required to proportion the wheel so that (η) will become a maximum. If (q) and (ν) are independent of the velocity (v), then (η) will manifestly be greatest when $(u.\cos.\alpha - v)v$ is a maximum. But this expression is the area of a rectangle which will be greatest when the figure is a square; or when:

$$v = \frac{u.\cos.\alpha}{2}.$$

Now as there are certain resistances which vary directly with the value of (v), it would be reasonable to infer that it would be more advantageous to make: $v < \frac{u}{2}$

Experience on the other hand has proved, however, that we should

rather increase the periphery velocity a little, and place the ratio:

$$\frac{v}{u} = 0{,}5 \text{ to } 0{,}55, \quad \text{or:} \quad v = 0{,}5\sqrt{2gH}.$$

By referring back for a moment, we will find that for the ordinary under shot percussion wheels: $v = 0{,}4\sqrt{2gH}$.

But we have for the velocity of entrance (u):

$$u = \sqrt{\frac{2gH}{1+\zeta}},$$

and hence:

$$\frac{v}{u} = 0{,}5\sqrt{1+\zeta};$$

and if we take $(\zeta) = 1/20 = 0{,}1$, we shall have approximately:

$$\frac{v}{u} = 0{,}5\,(1 + \tfrac{1}{20}) = 0{,}525$$

The expression $(1+1/20)$ is found by developing the radical and taking only the two first terms in the series and neglecting the remainder.

By inverting this ratio, and developing the radical into a series as before, the exponent becoming in this case however $=(-1/2)$, we have:

$$\frac{u}{v} = 2(1 - \tfrac{1}{20}) = 1{,}9.$$

The last expression for the coefficient (η), eq.(3) on the preceding page may be written thus:

$$\eta = (1-q)\frac{2v^2}{gH}\left(\frac{u}{v}\cos.\alpha - 1\right) - \vartheta$$

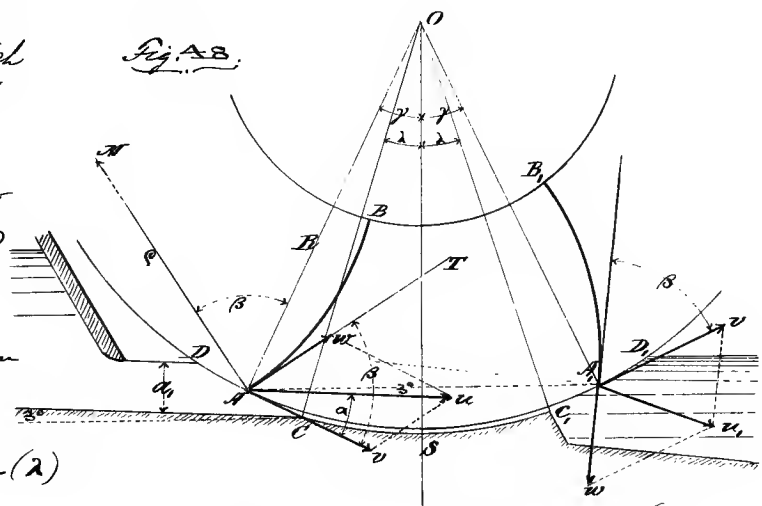
But from the above value for the velocity (v), we obtain: $v^2 = \frac{gH}{2}$; and if we now substitute the values of (v^2) and $(\frac{u}{v})$ in the last expression for (η) we shall get: $\eta = (1-q)(1{,}9\cos.\alpha - 1) - \vartheta$

Condition equations. To properly construct a Poncelet's Wheel, the following equations of condition must be observed. For the First, we have:

$$w.\sin.(\beta - \alpha) = v.\sin.\beta. \qquad (1).$$

(2°) - Let (2λ) be angle which the circularly hollowed out trough CC_1 covers at the circumference of the wheel. This is usually made so that it is symmetrically situated relatively to the vertical OS; and the arc CSC_1 usually is equal to the space between two buckets measured on the wheel's circumference, and hence we may regard the angle (λ) as a known quantity.

Fig. 48.



But there exists a certain relation between the angles (γ) and (λ). We have from the construction: $\gamma = \alpha + 3° = \lambda + \frac{i}{2R}$, where $(\frac{i}{2})$ is the half CA of the entrance arc CAD of the water $= C_1D_1$.

This arc (i) is small enough to be considered as a straight line, and hence we may place approximately:

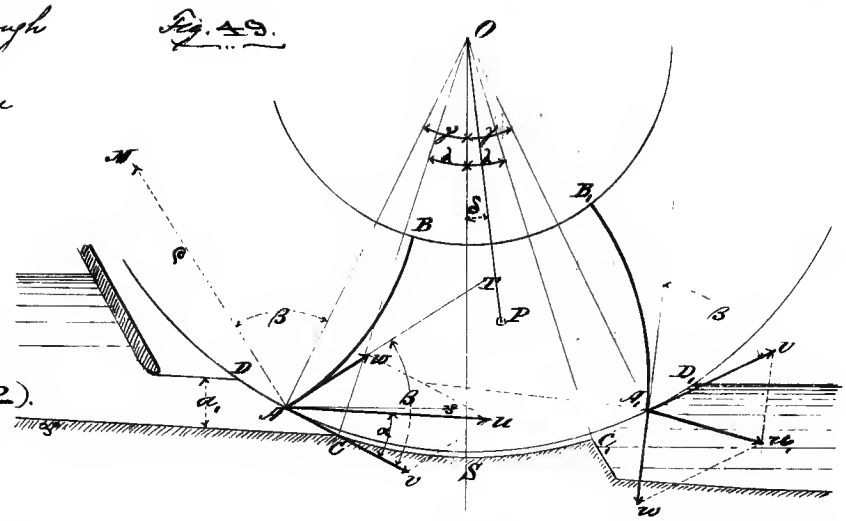
$$i = \frac{a_1}{\sin.\alpha},$$

where (a_1) denotes the depth of the water just in front of the wheel. We thus have:

$$\gamma = \lambda + \frac{a_1}{2R.\sin.\alpha} \quad \ldots\ldots (2).$$

which is the second Equation of condition.

Fig. 49.



Depth of the Wheel, (a). — (35) — Conceive an isolated particle of water flowing into the wheel at the point A. Let (P) be the point in its course thro' the wheel where it is in a momentary state of rest relatively to the wheel, and has just finished its ascent up the surface of the bucket, and is about to begin its descent. Denote the distance from O to P by (r), and the angle POS by (δ), which from the nature of the case is always less than (γ). This distance (r) upon which the depth (a) of the wheel depends, must now be calculated.

The living force with which the particle of water enters the wheel is

$$= \frac{w^2}{2} \text{ per unit of mass}$$

this expression is also Equal to the weight.

The total height to which this particle is raised, is equal to the projection of OA upon the vertical OS minus the projection of OP upon OS, or:

$$\text{height} = R.\cos.\gamma - r.\cos.\delta$$

Again, the centrifugal force acting upon a unit of mass at a distance (x) from the centre of rotation is $= \omega^2.x$, where (ω) is the angular velocity; but since: $\omega = \frac{v}{R}$, in the wheel, this expression also becomes:

$$\omega^2.x = \frac{v^2}{R^2}.x.$$

The work which the centrifugal force per unit of mass for each element of ~~force~~ between the limits (r) and (R).

Hence we have for the work

$$\text{work of centrifugal force} = \int_r^R \frac{v^2}{R^2}.x\,dx,$$

and for the total work Expended in raising a unit of water up to the above height we have:

$$\frac{w^2}{2} = g(R.\cos.\gamma - r.\cos.\delta) + \int_r^R \frac{v^2}{R^2}.x\,dx,$$

or:

$$w^2 = 2g(R.\cos.\gamma - r.\cos.\delta) + \frac{v^2(R^2 - r^2)}{R^2}$$

Transposing, we have:

$$w^2 - v^2 = 2gR\left(\cos.\gamma - \frac{r}{R}.\cos.\delta\right) - \frac{v^2.r^2}{R^2}$$

But the parallelogram of forces, in the figure, we have:

$$w^2 = u^2 + v^2 - 2u.v.\cos.\alpha\ ;$$

substituting this value of (w^2), and then dividing the equation thus by (v^2), we get:

$$\frac{u(u-2v.\cos.\alpha)}{v^2} = \frac{2g.R}{v^2}\left(\cos.\gamma - \frac{r}{R}.\cos.\delta\right) - \frac{r^2}{R^2}\ .$$

This last is a quadratic equation with respect to the ratio $\left(\frac{r}{R}\right)$, and may be written thus:

$$\frac{r^2}{R^2} + \frac{2g.R}{v^2}.\cos.\delta.\frac{r}{R} = \frac{2g.R}{v^2}.\cos.\gamma - \frac{u}{v}\left(\frac{u}{v} - 2.\cos.\alpha\right)$$

Substituting the value of (v) found above, as also that of the ratio $\left(\frac{u}{v}\right)$ on page 65, we have

$$\frac{r^2}{R^2} + \frac{4R}{H}.\cos.\delta.\frac{r}{R} = 4.\frac{R}{H}.\cos.\gamma - 1{,}9(1{,}9 - 2.\cos.\alpha)\ .$$

But since the angle (δ) is very small, we may place without great error $\cos.\delta = 1$; and as the angle (γ) is only 3° greater than the angle (α), we may place: $\cos.\gamma = \cos.\alpha$; and since these errors are made in both members, the two will balance each other. Singly, these errors would be too great to be admitted. We accordingly have:

$$\frac{r^2}{R^2} + 4.\frac{R}{H}.\frac{r}{R} = \left(4.\frac{R}{H} + 3{,}8\right)\cos.\alpha - 3{,}6\ .$$

Solving this equation, we obtain our third equation of condition:

$$\frac{r}{R} = -2\frac{R}{H} + \sqrt{4.\frac{R^2}{H^2} + \left(4.\frac{R}{H} + 3{,}8\right)\cos.\alpha - 3{,}6}\ .\ \ldots\ldots\ldots\ (3).$$

The expression under the radical sign in this equation is always positive, and the plus (+) sign alone has here any significance before it. The whole expression refers to a particle of water (... of volume ... of mass) entering the wheel at the point A.

When we consider a particle of water on the surface flowing into the wheel at the highest point D of the entrance arc (i), then the angles (γ) and (α) will be increased, and hence their cosines diminished, and thereby also the ratio $\left(\frac{r}{R}\right)$ diminished. Conversely, it may be shown in the same manner that were (C) considered as the point of entrance of the particle of water, the ratio $\left(\frac{r}{R}\right)$ would be increased.

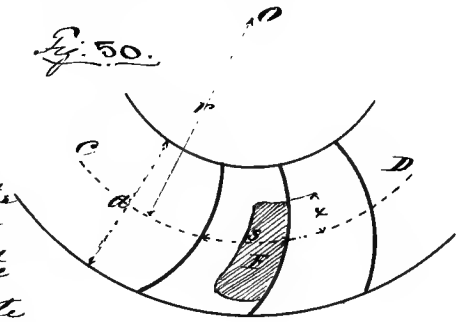
But as the particles of water do not act singly, but together, the first particles after they have almost reached their full height of ascent receive a fresh impulse from those which follow; the former will attain a greater height or a point nearer to the centre O of the wheel than is given by equation (3), while the last particles which enter the cell will not ascend so far on account of the return or downward flow of those first particles, which have meanwhile arrived at a state of rest relatively to the wheel, and commenced their descent, and thus prevent the last from reaching the height to which they would ascend if they were not thus hindered.

From these considerations, we shall evidently not depart too

far from the truth. if we assume that the entire body or volume of water, which rushes into the cell, simultaneously arrives at a state of relative rest, and begins its descent likewise.

Accordingly, if with (r) as a radius and O as centre, we describe a cylindrical surface, this latter will be the limit or boundary to the ascent of the water particles in the wheel.

Fig. 50.



Let F = the cross section of the volume of water taken into a bucket, drawn with the radius (r) = the distance from the centre (O) to the point where a particle of water arrives at a state of rest relatively to the wheel the dotted curve CD; also denote the distance measured on this arc between two consecutive buckets by (s), and the distance, in a radial direction, above this arc to which the foremost particles of water are driven by the impulse of the following particles, by (x)

Evidently we have then:

$$x > \frac{F}{2s}, \text{ from which then follows:}$$

$$a > R - r + \frac{F}{2s}$$

But this area F may be expressed by: $F = a_1 \cdot u \cdot \frac{\varepsilon}{v}$,

and (s) by: $s = \frac{r}{R} \cdot e$

Hence we have:

$$a > R - r + \frac{a_1}{2} \cdot \frac{u}{v} \cdot \frac{R}{r}.$$

If we now insert here the values of $(\frac{u}{v})$ found above: viz: $\frac{u}{v} = 1,9$, we obtain: $a > R - r + 0,95 \cdot a_1 \frac{R}{r}$

In practice, however, we generally make:

$$a > R\left(1 - \frac{r}{R}\right) + a_1 \cdot \frac{R}{r}, \quad \text{(A)}.$$

in which formula the value of $(\frac{R}{r})$ is to be taken from Equation (3.) above.

Radius of curvature. (ρ). — We must now find some condition equation for the mean radius of curvature of the buckets. First however, it is necessary to recall a few principles from Mechanics.

When a material point is constrained to move in a given direction or along a given surface, let (K) = the force applied to the point which causes it to move in the prescribed direction, and (N) = the normal pressure of the point upon the surface, and (−N) = the reaction of the latter. The point will move freely if we conceive only

the two forces (K) and $(-N)$ to act upon it, and hence its motion may be regarded as produced by the resultant of (K) and $(-N)$.
Let us next resolve this resultant into two components, one of which is the centripetal force, which will be denoted by (P), and the other a tangential force to the surface of the bucket, which we will call (T).
We may accordingly place:

Resultant of $(K, -N)$ = Resultant of $(P$ and $T)$.

These forces must balance, since equilibrium exists between the forces K, N, P, and $-T$, where the centrifugal force (P) is used in place of the centripetal force (P).
But when four forces balance, any one of them can be regarded as the resultant of the other three; and we may therefore place:

N = Resultant of $(K, P, \&\ -T)$

Let us next resolve these forces in the parenthesis into two components each, one normal to the surface, and the other tangential. We thus have: K = Result. of K_n and K_t;
P = " " " P_n " P_t, and
$-T = -T$, as this can not be resolved being already tangential.
Now since these forces balanced each other before resolution, those of their components which lie in the same plane or surface must now balance also, that is the components K_t, P_t, and $-T$ must be in equilibrium. The components (of the forces) which are normal to the surface must in like manner also balance, and hence we have:

$N = P_n \pm K_n$, according as K_n is directed against the convex or concave surface of the bucket.

We have, however, tacitly assumed in the foregoing that the surface in question is at rest. When on the other hand the surface is also in motion, then we must regard (K) as the relative moving force, which may be determined in the following manner.
Let (S) be the system of forces in motion. The elementary motions of this system or the motion during an element (dt) of time may be resolved into a motion of rotation and a motion of translation.
Let (A) be the position of the momentary axis for the rotary motion during the indefinitely small space of time (dt); and
let (ω) be the corresponding angular velocity.
The direction of the momentary axis (A) and the angular velocity (ω) is however the same in all cases; the position of the axis (A) alone is changeable.
Further let (p) be the material point in question, and
(q) that point in the system of forces (S) which coincides with the point (p) at the moment of time under consideration.
Let (Φ) = the momentary acceleration of the point (q) at the moment of its coincidence with the point (P);
(w) = the relative velocity of the point (P) to the point (Q), and
(v) = the absolute velocity of " " " P.
Now conceive the relative velocity (w) to be projected upon the plane

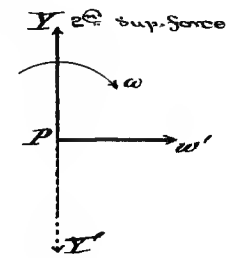
which is perpendicular to the axes (A), (the plane of the paper), and denote by (w') this projection.

The first supplementary force (ergänzungs-kraft) imparts the acceleration (φ) to the point (q), but in an opposite direction.

The second supplementary force imparts the acceleration (2ω·w') to the point P, but this force is not yet determined in direction.

The sense of this second supplementary force is however taken so that if its direction were turned through an angle of 90°, it would fall in the direction of (w'), the rotation or turning being considered as taking place in the sense of (ω), the angular velocity.

If we assume that the direction of the angular velocity (ω) is that of the hands of a watch, then PY would be the 2d supplementary force; because if it were turned through an angle of 90°, its direction would coincide with that of the projection of the relative velocity (w), whereas if its direction were assumed to be that of PY', this latter would have to be turned through an angle of 270° to fall in the direction of (w').



The case of our Poncelet wheel is precisely similar to the above case of motion. The surface upon which the material point (the particle of water) is constrained to move, is here cylindrical, (or more generally speaking, a curved surface,) and hence the path described by such a point may be considered as circular.

In the case of the wheel we thus have:

$$F_n = F, \text{ simply, and hence: } N = F \pm K_m.$$

The absolute moving force is here gravity.

The first supplementary force is the [illegible] of the point Q in the bucket with which the point P coincides.

The second supplementary force is

The angular velocity (ω) also is here constant, being that of the wheel itself; and the projection (w') of the relative velocity (w) is here identical with (w) since the wheel turns about its geometrical axis, and this latter is perpendicular to the plane of our section upon which (w) is conceived to be projected; but as this sectional plane is that in which (w) itself lies, its projection (w') is thus = (w). We also have:

$$\omega = \frac{v}{R}, \text{ from its definition.}$$

We thus see that while the water particle moves up the surface of the bucket, no normal pressure can be exerted against it which would cause it to leave this surface and move in the air alone; in its descent, however, this may happen when the bucket has an

proper form. In Fig. 51, let SL be any position of a bucket while a particle of water (P) is moving up along its surface, and let (P) be the position of the particle at any period of its upward course.

The force of gravity is here (g) and has two components one in the direction of the rad. is OP ... and the other ...

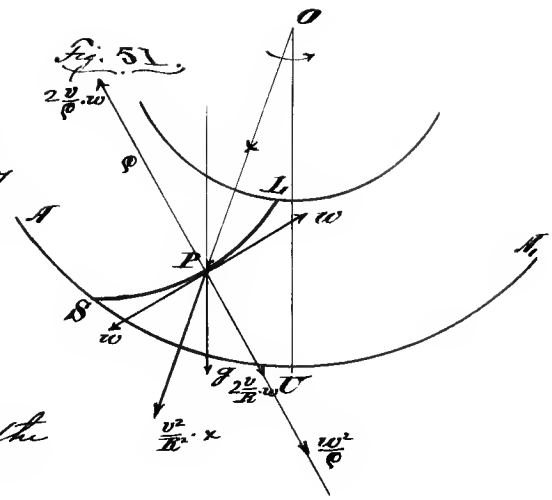


Fig. 51.

Let (x) be the distance (OP) from the centre of the wheel to the point P

When the particle of water begins its descent, the direction of (w) changes, and therefore also that of the second supplementary force. The curvature of the buckets must accordingly be such that no opportunity will be afforded to the water particle to leave the surface of the bucket and fall freely thro' the air.

To ascertain the proper form for the buckets, and hence also to determine the radii of curvature for the various elements of the surface, it will be sufficient to investigate the motion of the water particle at the beginning and the end of its descent or downward course.

Let us suppose that when the water particle has arrived at a state of rest relatively to the wheel, viz: at the point P_0, Fig. 52, the edge of the bucket along the surface of which it moves coincides with the point U in the vertical radius. Also let us take a mean radius of curvature (ρ); and from the point M as centre, describe the circular bucket UL', so that the tangent to the curve at the point L shall be vertical, the line ML being drawn horizontal. Denote the angle MUN by (β).

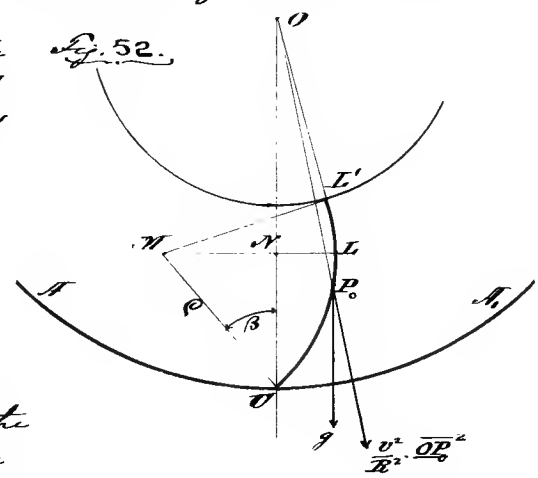


Fig. 52.

At the moment that the water particle has arrived at a state of relative rest, the moving force as well as the second supplementary force both become = 0 = zero.

Now in order that the water particle shall not tend to leave the surface of the bucket, it is only necessary that the point (P_0) shall be lower than the point L', as MLU is drawn as a right angle; and it will therefore be more than sufficient if the point (P_0) is

made to lie even lower than the point L, that is, if the length:

$$OL < OP_0$$

But from the construction of the figure:

$$OL = \sqrt{\overline{OT}^2 + \overline{TL}^2} < r'',$$

and: $\overline{OT}^2 = (R - \rho . \cos.\beta)^2$,

and: $\overline{TL}^2 = \rho^2 (1 - \sin.\beta)^2$;

hence we have:

$$(R - \rho . \cos.\beta)^2 + \rho^2 (1 - \sin.\beta)^2 < r''^2,$$

or: $$R^2 - 2.R.\rho.\cos.\beta + 2\rho^2 (1 - \sin.\beta) < r''^2.$$

Dividing through by $(2R^2)$, we shall obtain the first equation of condition for (ρ) the radius of curvature of the buckets, and the fifth general equation of condition for a Poncelet wheel.

$$\left(\frac{\rho}{R}\right)^2 . (1 - \sin.\beta) - \frac{\rho}{R} . \cos.\beta + \tfrac{1}{2}\left(1 - \frac{r''^2}{R^2}\right) < 0 \text{ (zero)} \quad \ldots\ldots\ldots (5).$$

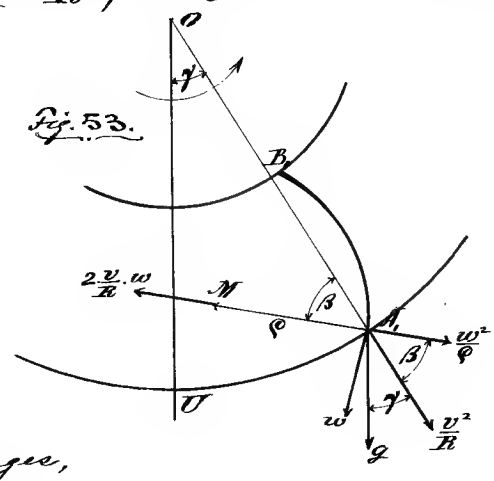


Fig. 53.

For the relations of the motion of the water particle as it discharges from the bucket after having finished its upward and downward motions in the wheel, we have the following forces acting, — the force of gravity (g), and the just supplementary force or centrifugal force. $\frac{v^2}{R^2} . R = \frac{v^2}{R}$; and if the water particle shall be pressed against the surface of the bucket as it discharges, we have this condition, that:

$$2 . \frac{v}{R} . w < \frac{w^2}{\rho} + \frac{v^2}{R} . \cos.\beta + g . \cos.(\beta + \gamma) \quad \ldots\ldots\ldots (1.)$$

But from our former parallelogram of forces, we have:

$$w^2 = u^2 + v^2 - 2.u.v.\cos.\alpha = \left(\frac{u^2}{v^2} + 1 - 2.\frac{u}{v} . \cos.\alpha\right) v^2;$$

and by substituting the above value of the ratio: $(\frac{u}{v})$

$$w^2 = (4{,}6 - 3{,}8 . \cos.\alpha)\, v^2 = \mu . v^2,$$

the coefficient (μ) being used for brevity's sake, as the expression it represents is constant.

Introducing this value of (w) in the above condition (1.), we obtain:

$$\frac{2\sqrt{\mu}}{R} . v^2 < \frac{\mu . v^2}{\rho} + \frac{v^2}{R} . \cos.\beta + g . \cos.(\beta + \gamma),$$

and multiplying thro' by (R):

$$2\sqrt{\mu} . < \mu . \frac{R}{\rho} + \cos.\beta + g . \frac{R}{v^2} . \cos.(\beta + \gamma)$$

But because $v^2 = \frac{gH}{2}$, we have: $g . \frac{R}{v^2} = 2 . \frac{R}{H}$; and if we insert this value in the last expression and transform the latter, we shall obtain for a second condition relatively to the radius of curvature:

$$\frac{R}{\rho} > \frac{1}{\mu}\left[2\sqrt{\mu} - \cos.\beta - 2 . \frac{R}{H} . \cos.(\beta + \gamma)\right], \quad \ldots\ldots\ldots (5a).$$

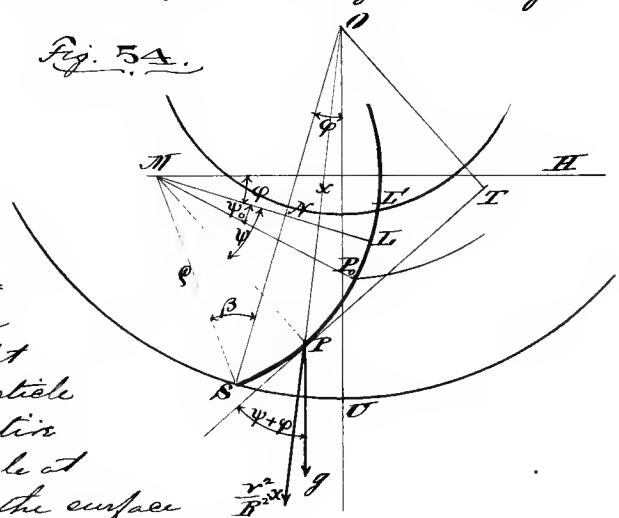
where $(\mu) = 4.6 - 3.8 . \cos . \alpha$.

An important condition, however, yet remains to be expressed analytically. We have hitherto assumed that the point of entrance (A) of the water particle, whose course through the wheel we have been tracing, is on the same level, or in the same horizontal line as the point (A_1) where it discharges from the wheel. This very important condition, that these two points shall be in the same horizontal line, we must now endeavor to express.

This condition may also be stated thus: that the time required for the water particle to complete its ascent and descent along the surface of the bucket, and in its relative motion, shall be equal to the time required by the wheel, or rather a point on the wheel's circumference, to describe or pass through the angle (2γ).

Relative Motion. —

Fig. 54.



In Fig. 54, let M be the centre of curvature; MH a horizontal line thro' M; OV the vertical radius; SL' a bucket, the edge S of which is at a point where the angle $SOV = \varphi < \gamma$; ML is perpendicular to the radius SO, and N is the point of intersection. Further let P_0 be the position of the water particle as it has attained a state of relative rest; P the position of the particle at any point in its ascent along the surface of the bucket; MP the radius of curvature at the point P, and $OP = x$ = the distance from the centre O to the point P; PT is the tangent to the curve of the bucket at the point P, and OT is drawn perpendicular to this tangent.

For the relative path of the water particle we have:

$$\text{par. } \overline{PT} = \frac{d^2 s}{dt^2}$$

or,

$$v = \rho . \psi$$

assuming [illegible] the [illegible] of

$$\frac{d^2 s}{dt^2} = \rho . \frac{d^2 \psi}{dt^2} \quad , \quad (\rho) \text{ being considered as constant.}$$

The relative motion of the water particle is therefore determined by means of the expression:

$$\rho . \frac{d^2 \psi}{dt^2} = g . \cos . (\varphi + \psi) + \frac{v^2}{R^2} \, x . \cos . \widehat{OPT}$$

But the expression

$$x . \cos . \widehat{OPT} = PT = NO . \cos . \psi + MN . \sin . \psi \; ;$$

and: $$NO = R - \rho . \cos . \beta \text{ , and } MN = \rho . \sin . \beta \; ;$$

whence: $$x . \cos . \widehat{OPT} = (R - \rho . \cos . \beta) \cos . \psi + \rho . \sin . \beta . \sin . \psi \; ,$$

and by substitution in the above equation:

$$\rho.\frac{d^2\psi}{dt^2} = g.\cos.(\varphi+\psi) + \frac{v^2}{R^2}\Big[(R-\rho.\cos.\beta)\cos.\psi + \rho.\sin.\beta.\sin\psi\Big]$$

Now the path described by the water particle during the time (dt) is $= -R.d\varphi$, which is the absolute value; but this path is also $= v.dt$, or: $v.dt = -R.d\varphi$,

from which there follows:

$$\frac{v^2}{R^2} = -\frac{d\varphi^2}{dt^2}, \text{ or } \frac{dt^2}{d\varphi^2} = -\frac{R^2}{v^2};$$

and by multiplying the above equation through by this value:

$$\rho.\frac{d^2\psi}{d\varphi^2} = g.\frac{R^2}{v^2}.\cos.(\varphi+\psi) + (R-\rho.\cos.\beta)\cos.\psi + \rho.\sin.\beta.\sin.\psi.$$

Strictly, we should be obliged to integrate twice in succession, as this is a differential equation of the second order; and therefore we should have to determine two constants of integration.

The first of these constants would be found from the fact that when the particle of water arrives at a state of relative rest, we would have the angle: $\psi = \psi_0$, and hence: $\frac{d\psi}{d\varphi} = 0 =$ zero.

The second constant is determined from the circumstance that:

$$\psi = \frac{\pi}{2} - \beta, \text{ and from } \varphi = \gamma$$

In this manner a finite equation between the angles (φ) and (ψ) may be found, in which all the constants are known; and this equation, moreover, must be true when we substitute for (φ) and (ψ) the values: $\psi = \frac{\pi}{2} - \beta$, and $\varphi = \gamma$.

This mode of procedure is however too difficult to be followed, and we accordingly adopt in general some method of approximation. To this purpose, let us assume that the points (S) and (T) in Fig. 54 coincide, or that $(\varphi) = 0$. The result of this is, that the value of $\cos.(\varphi+\psi)$, and that of the acceleration in the direction of TI, will become too great, while the acceleration in the downward direction becomes too small. This assumption that $(\varphi) = 0$ leads however to so small an error, that it may be entirely neglected.

We thus have:

$$\rho.\frac{d^2\psi}{dt^2} = g.\cos.\psi + \frac{v^2}{R^2}(R-\rho.\cos.\beta)\cos.\psi + \frac{v^2}{R^2}.\rho.\sin.\beta.\sin.\psi,$$

and:

$$\frac{d^2\psi}{dt^2} = \frac{v^2}{R^2}\left(\frac{g.R^2}{\rho.v^2} + \frac{R}{\rho} - \cos.\beta\right)\cos.\psi + \frac{v^2}{R^2}.\sin.\beta.\sin.\psi;$$

This last equation may be put under the form of:

$$\frac{d^2\psi}{dt^2} = \frac{A}{2}.\cos.\psi + \frac{B}{2}.\sin.\psi.$$

In order to differentiate this last equation, we must multiply it by the identical equation:

$$2.\frac{d\psi}{dt}\,dt = 2.d\psi;$$

we thus obtain:

$$d\left(\frac{d\psi}{dt}\right)^2 = A.d(\sin.\psi) - B.d(\cos.\psi)$$

The two members of this equation are now perfect differentials, and may therefore be integrated without any difficulty. Accordingly, if we integrate on the right side or second member between the limits (ψ_0) and (ψ), and on the left side or first member between the limits (0) and $\left(\frac{d\psi}{dt}\right)$, we obtain:

$$\left(\frac{d\psi}{dt}\right)^2 = A(\sin.\psi - \sin.\psi_0) + B(\cos.\psi - \cos.\psi_0),$$

whence:

$$dt = \frac{d\psi}{\sqrt{A(\sin.\psi - \sin.\psi_0) + B(\cos.\psi - \cos.\psi_0)}},$$

and:

$$t = \int_{\psi_0}^{\frac{\pi}{2}-\beta} \frac{d\psi}{\sqrt{A(\sin.\psi - \sin.\psi_0) + B(\cos.\psi - \cos.\psi_0)}},$$

which is the expression for <u>one-half</u> the time occupied by the upward and downward motion of the water particle along the surface of the bucket.

With respect to the coefficients (A) and (B), we have from the above: $A = \frac{v^2}{R^2} m$, and: $B = \frac{v^2}{R^2} 2.\sin.\beta$,

where (m) denotes for simplicity's sake:

$$m = 2.\frac{R}{\varphi}\left(2\frac{R}{v^2} + 1\right) - 2.\cos.\beta;$$

but since $\left(\frac{g}{v^2}\right) = \frac{2}{H}$, the value of (m) may also be written

$$m = 2.\frac{R}{\varphi}\left(2.\frac{R}{H} + 1\right) - 2.\cos.\beta$$

Substituting these values of the coefficients (A) and (B), the above integral thus becomes:

$$t = \frac{R}{v}\int_{\psi_0}^{\frac{\pi}{2}-\beta} \frac{d\psi}{\sqrt{m(\sin.\psi - \sin.\psi_0) + 2.\sin.\beta(\cos.\psi_0 - \cos.\psi)}}.$$

This <u>half time</u> (t) must however be:

$$t = \frac{R}{v}.\gamma,$$

which is the time in which the point on the wheel's circumference, at the edge of the bucket, describes the arc corresponding to the angle (γ).

We thus have for the condition that the water shall discharge at the same level as it entered the wheel, the following equation:

$$\gamma = \int_{\psi_0}^{\frac{\pi}{2}-\beta} \frac{d\psi}{f(\psi)}, \qquad (6).$$

where: $f(\psi) = \sqrt{m(\sin.\psi - \sin.\psi_0) + 2.\sin.\beta(\cos.\psi_0 - \cos.\psi)}$,

and:

$$m = 2.\frac{R}{\varphi}\left(2\frac{R}{H} + 1\right) - 2.\cos.\beta.$$

Let us now see what follows when we make: $R = 2H$;

$a_1 = \frac{1}{16}.H$; $\alpha = 15°$; $\gamma = 18°$; and if the cylindrical hollowing-out of the trough extends over the space, on the wheel's circumference, of <u>ten cells</u>, then: $\lambda = 18° - \frac{1}{2x.\sin.\alpha} \cdot \frac{180°}{\pi} = 9°$;

from this there now follows the number of buckets:

$$z = \frac{360^\circ}{9^\circ} = 40$$

We also have:

$$\frac{\sin.(\beta-\alpha)}{\sin.\beta} = \cos.\alpha - \operatorname{ctg}.\beta.\sin.\alpha = \frac{v}{w},$$

whence:

$$\operatorname{ctg}.\beta = \frac{\cos.\alpha - \frac{v}{w}}{\sin.\alpha};$$

from this follows, after substituting the values of $\left(\frac{v}{w}\right)$ and (α):

$$\beta = 30^\circ\ 25',$$ for which we may take in round numbers:

$$\beta = 30^\circ.$$

The value of the ratio $\left(\frac{r}{R}\right)$ may now also be found, which is:

$$\frac{r}{R} = 0{,}876$$

Dividing equation of condition (4) by H, we obtain:

$$\frac{a}{H} > 2.0{,}124 + \frac{1}{6.0{,}876}, \text{ or: } \frac{a}{H} > 0{,}44.$$

This ratio however should be taken somewhat greater on account of the irregularities and splashing of the water; and we therefore may take one and one-half as great in order to be certain that the depth (a) will become great enough. Hence we may place.

$$\frac{a}{H} = 0{,}66, \text{ or: } a = \tfrac{2}{3}H$$

From the condition (5) there also follows the value of the ratio: $\frac{\varrho}{R}$:

$$\frac{\varrho}{R} > 0{,}147.$$

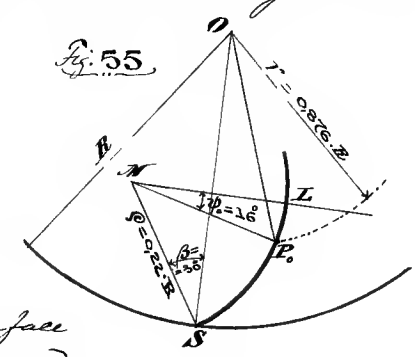
Equation of condition (6) is also dependent upon the condition (5), or the condition for the curvature of the buckets. Let us assume a value for the ratio $\left(\frac{\varrho}{R}\right)$ one and one-half $(1\frac{1}{2})$ times greater than the above limiting value, and then ascertain whether equation of condition (6) can be satisfied by the substitution of this increased value.

Assuming therefore: $\frac{\varrho}{R} = 0{,}22$, there then follows:

$$m = \frac{2.5}{0{,}22} - 2.\cos.30^\circ = 43.72$$

We must next find the value of (ψ_0), which is best done by construction. With the radius (R) draw any arc, and any radius OS; then draw the line $SM = \varrho = 0{,}22.R$ making an angle $\beta = 30^\circ$ with OS; from the point M as centre, describe the circular arc of the bucket SL, and also draw the line ML perpendicular to OS; finally from O as centre with the radius $r = 0{,}876.R$, draw the arc which limits the ascent of the water particles on the surface of the buckets, and which cuts SL in P_0, and draw the radius MP_0. The angle LMP_0, which we have named (ψ_0) will then be found to be:

Fig. 55



$$\psi_0 = 16^\circ$$

We may now find the value of $f(f(\psi))$ for which we have:

$$f(\psi) = \sqrt{43,72 \cdot \sin.\psi - \cos.\psi - 11,09}\ ;$$

as also that of the integral:

$$\int_{16°}^{60°} \frac{d\psi}{f(\psi)}$$

This integral is best calculated by means of some of the methods of mechanical quadrature, as for instance, "Simpson's Rule". We may put the above integral under the form:

$$\int_{x_0}^{x_n} f(x)\,dx$$ Let us then divide the difference between (x_0) and (x_n) into (n) equal parts, so that:

$$x_1 - x_0 = x_2 - x_1 = x_3 - x_2 = \cdots\cdots = x_n - x_{n-1}\ ,$$

and:

$$\int_{x_0}^{x_n} f(x)\,dx = (x_n - x_0)\left\{ K_0 \cdot f(x_0) + K_1 f(x_1) + \cdots + K_n f(x_n) \right\},$$

where:

$$K_1 + K_2 + K_3 + \cdots\cdots + K_n = 1$$

and:

$$K_0 = K_n\ ;\ K_1 = K_{n-1}\ ;\ \text{etc}$$

Let us divide the integral into two parts, each of which may be integrated separately:

$$\int_{16°}^{60°} \frac{d\psi}{f(\psi)} = \int_{16°}^{24°} \frac{d\psi}{f(\psi)} + \frac{36}{90}\left[\frac{7}{f(24°)} + \frac{32}{f(33°)} + \frac{12}{f(42°)} + \frac{32}{f(51°)} + \frac{7}{f(60°)}\right],$$

whence:

$$\gamma = \int_{16°}^{24°} \frac{d\psi}{f(\psi)} + 9,29°$$

We must now ascertain whether the value of the first term in this expression for the angle (γ) will be sufficiently great so that the sum of the two terms will become $= 18°$, which was the value assigned in the outset to the angle (γ)

Let us place:

$$\psi = \psi_0 + \Delta\psi\ ,$$

as the difference between the two angles (ψ) and (ψ_0) is here only $(8°)$; for this small angle of $(8°)$ we may place approximately:

$$\sin.(\Delta\psi) = \Delta\psi\ ,\ \text{itself},\ \text{and}\ \cos.(\Delta\psi) = 1\ ;$$

and hence also: $\sin.\psi = \sin.\psi_0 + \cos.\psi_0 \cdot \Delta\psi$,

and: $\cos.\psi = \cos.\psi_0 - \sin.\psi_0 \cdot \Delta\psi$;

from this we now have:

$$f(\psi) = \sqrt{m \cdot \cos.\psi_0 + \sin.\psi_0 \cdot \Delta\psi}\ ,$$

and hence:

$$\int_{\psi_0}^{\psi_0+\Delta\psi} \frac{d\psi}{f(\psi)} = \frac{1}{\sqrt{m\cdot\cos.\psi_0 + \sin.\psi_0}} \cdot \int_{\psi_0}^{\psi+\Delta\psi} \frac{d\psi}{\sqrt{\Delta\psi}} = \frac{2}{\sqrt{m(\cos.\psi_0) + \sin.\psi_0}} \int_0^{\Delta\psi} \frac{\frac{1}{2}d\psi}{\sqrt{\Delta\psi}}.$$

The integral in the second member of the last equation is the integral of the differential of $\sqrt{\Delta\psi}$; that is: $d(\sqrt{\Delta\psi}) = \frac{\frac{1}{2}d\psi}{\sqrt{\Delta\psi}}$;

hence there follows finally:

$$\gamma = \int_{16°}^{24°} \frac{d\psi}{f(\psi)} + 9,29° = \frac{2\sqrt{\Delta\psi}}{\sqrt{m(\cos.\psi_0) + \sin.\psi_0}} + 9,29°,$$

and by substituting the values found above:

$$\gamma = 6,59° + 9,29° = 15,88°$$

This value of (γ) is not equal to $(18°)$, and hence we must increase the ratio $(\frac{\rho}{R})$. Let us place:

$$\frac{\rho}{R} = 0,25 \text{ , and therefore: } \rho = \tfrac{1}{2}H$$

By a new construction like Fig. 55, we obtain for the value of the angle (ψ_0): $\psi_0 = 20°$;
and by repeating the above calculation with this new value of (ψ_0), we shall finally obtain a value for the integral which will cause the above expression for (γ) to become very nearly equal to $(18°)$.

We have thus found for the best relations between the elements of a Poncelet's Wheel, the following values:

$$R = 2H \text{ ;} \qquad a = \tfrac{2}{3}\cdot H \text{ ;}$$
$$\rho = \tfrac{1}{2}H \text{ ;} \qquad a_1 = \tfrac{1}{6}\cdot H \text{ ;}$$
$$\alpha = 15° \text{ ;} \qquad \beta = 30° \text{ ;}$$
$$\gamma = 18° \text{ ;} \qquad v = 0,5\sqrt{2g.H}$$

We also found for the working co-efficient (η) of these wheels, the expression: $\eta = (1-q)(1,9\cdot\cos.\alpha - 1) - \vartheta$,
where: $q = \frac{s}{a_1} = \frac{6,5}{H} = \frac{0,09}{H}$,
the value of (s) being:

$$s = 0,015.$$

We thus have for: $H = 0,75^{m}$:

$$\eta = 0,735 - \vartheta \text{ ;}$$

and for: $H = 1,5^{m}$

$$\eta = 0,785 - \vartheta \text{ ;}$$

and if we place the term (ϑ) in which are included all those losses which can not be made subjects for mathematical investigations: $\vartheta = 0,105$,
we shall obtain:

$$\eta = 0,63 \text{ to } 0,68 \text{ , or about } \tfrac{2}{3}.$$

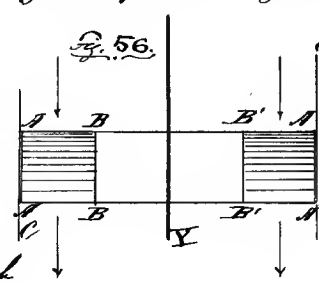
Turbines.

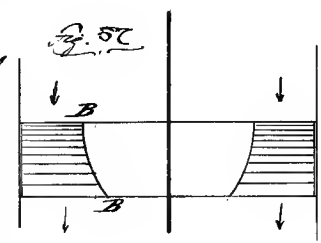
The essential difference between Turbines and the ordinary water wheels is that in the former the water is discharged at a different point from that at which it entered the wheel, and flowe through the rim of the wheel also. In Turbines, the buckets are always curved surfaces, and should be so constructed that the curvature is continuous instead of changing its direction suddenly.

In the calculation of turbines, the thickness of the buckets can not be neglected as was done in the case of the ordinary vertical water wheels, because the number of the buckets is great relatively to the size of the turbine

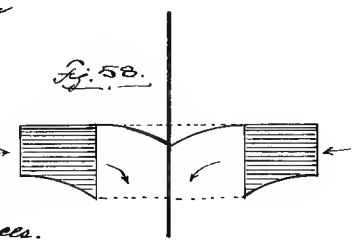
According as the water enters the wheel, Turbines are divided into: (1°):— Inner-shot Turbines, in which the water enters from the centre, and discharges at the circumference; —(2°):— Outer-shot Turbines, where the water enters at the outer circumference and is discharged towards the centre; and (3°):— Side-shot Turbines, in which the water enters at the upper bounding surface of the horizontal wheel and is discharged at the lower surface.

In Fig. 56 is represented a side shot turbine, in which the upper and lower surfaces are planes, the portion BB' being closed and the rest open to admit the passage of the water. The inner cylindrical surface BB is closed and fixed to the wheels, while the outer AA is open but more as close as possible to a stationary side wall, such as a tube, etc. The water enters and flowe out in the direction of the arrows. At the points in the wheel where the entrance and discharge of the water occurs, the cross-section of the rim (Krang) is bounded by straight lines AB, B'A', Figs. 56 and 57, parallel to each other; the side of the cross section, AA is always parallel to the vertical axis Y, while the side BB may be either parallel to AA, or inclined to the horizontal ABA', or even a curve, as BB, Fig. 57.





For an outer-shot turbine, the cross section of the rim may have a form like Fig. 58.



For Figs. 56 and 57, the entrance and outlet surfaces are parallel planes. For Fig. 58, on the other hand, they are concentric cylindrical surfaces.

The less the velocity of the water as it leaves the wheel, the greater the work absorbed by the machine, and the less is also the loss due to the living force of the discharging water. On this account the curvature of the buckets

should therefore be such that the greatest amount of work shall be taken up and in turn given out by the wheel.

In turbines, the water may enter either along the entire mantle in the case of an inner or outer-shot turbine, or the entire plane upper surface in the case of a side-shot turbine, in which events these wheels are called "full turbines"; or it may enter at only a portion of this mantle, when they are termed "partial" turbines.

Owing to fluctuations in the quantity of water which supplies the wheel, a "full" turbine may at times become virtually a partially filled one, which however does not affect the action and construction of the motor. A partial turbine on the other hand, can not be used as a full turbine.

Turbines may act either in the air, or submerged under water; but the effect or work performed by the wheel when the water is discharged into the air merely is considerably different from that performed by a submerged turbine which is similar to the first in all other respects.

A turbine usually consists of two wheels or rather rims, each of which is provided with buckets. One of these is fixed, and is termed the "guide-wheel" and its buckets the "guide buckets", while the other is fixed to the vertical axis and is movable, in imparting by its rotation motion to the machinery; this movable wheel is termed the "moving-wheel" (Lauf-rad) and its buckets the moving-wheel buckets.

When the turbine is enclosed in a tube, in which case it is always side-shot, it is called a "tube turbine" (Rohr-turbine); the tube serves to convey, as well as to carry off the water.

In an outer shot turbine, the water after having performed its work may be conducted off through a tube, as in Fig. 59.

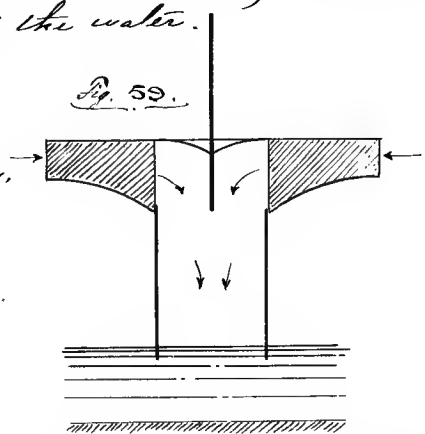


Fig. 59.

The tube may be made of iron or masonry, and the turbine [illegible] from 10 metres above the surface of the lower water.

The theory of turbines is capable of a much more accurate mathematical investigation and development than that of the ordinary vertical water wheels in consequence of the far greater uniformity in the action of the water. The theory may at first be developed in a perfectly general manner, and afterwards applied to the various cases which present themselves. It deals also with those rules for proportioning the several elements entering into the construction of a turbine, by which the greatest possible working coefficient will result.

without first having determined upon this co-efficient.

Let: Q = the quantity of water flowing to the turbine in one second;
H = -"- height or distance between the upper and lower water surfaces, or in other words, the head;
$E_0 = 1000 . Q . H$ = the absolute or theoretical effect;
$N_0 = \frac{E_0}{75}$ = the absolute effect in horse-powers;

$E = \eta . E_0$ = the useful effect or work performed by the wheel;
$N = \eta . N_0$ = -"- -"- -"- -"- in horse powers;
and: η = the working coefficient.

This coefficient (η) depends: (1°):— upon the friction and impact of the water in the wheel; (2°):— upon the loss of water, and hence also of work, due to the several play-spaces; and (3°):— upon the loss of work due to the axle-friction. The vertical axles in these wheels are quite large, and as they have a great angular velocity, a considerable amount of work is expended in overcoming the friction.

We may accordingly express the coefficient (η) as a function of three magnitudes, which depend upon the above three causes of loss of work. Thus let:

ε = the working coefficient of the wheel when the various hydraulic resistances alone are taken into consideration;
$\varepsilon . H$, will then be the useful head, (wirksame gefälle).

Further, let:
φ = the working coefficient due to the loss of water through the play spaces per second, whereon
φQ = the loss of water due to these play spaces.

Also let
μE_0 = the loss of work due to the axle friction.

From these three coefficients (ε, φ, and μ), the general working coefficient (η) may now be found in the following manner.

We have here:

$$\frac{1}{E_0}\left(1000 . \varphi Q . \varepsilon H - \mu . E_0\right) = \eta = \varphi\varepsilon - \mu ,$$

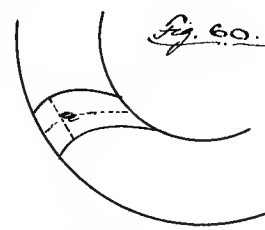
by which the value of (η) has been expressed in terms of the coefficients relating to the above three losses of work separately.

Further let:
r_1 = the radius of the cylindrical entrance surface at which the water enters the wheel; and:
r_2 = the radius of the cylindrical surface at which the water is discharged from the wheel

For inner-shot turbines, $r_1 < r_2$; while in outer-shot turbines, $r_1 > r_2$. For side-shot wheels; where the water enters from a plane surface, r_1 and r_2 denote mean values of the radii drawn to the outer and inner bounding cylindrical surface at the upper and lower side of the turbine respectively.

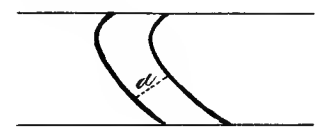
Also let: z_0 = the number of guide channels (leit canäle) in the guide-wheel; and:
z = the number of channels in the moving-wheel.
In the case of full turbines, (z_0) is the number of buckets in the guide-wheel, and (z) likewise the number of buckets in the moving wheel.
Again let: δ_0 = the thickness of the buckets at that point in the guide-apparatus or wheel where the water enters the moving wheel;
δ_1 = the thickness of the buckets in the moving wheel at the point of entrance of the water; and:
δ_2 = the thickness of the buckets in the moving wheel at the point of discharge of the water.
The middle line of the two canals, one in the guide and the other in the moving wheel, may in general be a curve of double curvature.
The cross-section of a channel in either the guide or moving wheel is formed by any arbitrary section whose plane is equally inclined to the two buckets and is parallel to the axis of the wheel, in the case of inner or outer-shot turbines.

Fig. 60.



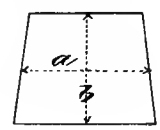
In inner-shot turbines, the cross-section is a trapezoid while in the other two cases, it is a rectangle.

Fig. 61.



Let (a) and (b) denote the mean dimensions of the cross-section whether in an inner or outer shot turbine.
Then (a) and (b) are the dimensions exclusive of the thickness of the buckets.
a = [illegible] of the canal
b = " inside [illegible]
In this sense a & b;

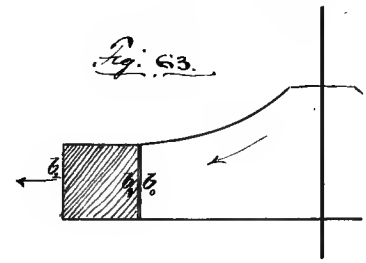
Fig. 62.



a_0, b_0 } = cross-sectional dimensions of the channels in the guide-wheel at the point where the water leaves this wheel and begins to enter the moving wheel;
a_1, b_1 } = the same dimensions of the channels in the moving wheel at the point of entrance of the water;
a_2, b_2 } = the same dimensions of the channels of the moving wheel at the point that the water begins to leave the wheel.

In all cases we must have $b_0 = b_1$, as will be seen from the figure.
Also let.
I_0 = the total area of all the cross-sections in the guide-wheel;

Fig. 63.



But this total area is:

$$F_0 = z_0 . a_0 . b_1$$

Further let:

F_1 = the total area of the cross sections of the channels in the moving wheel at the points of entrance of the water;

hence: $F_1 = z . a_1 b_1$;

and: F_2 = the total area of the cross sections of the canals in the moving wheel at the points of discharge of the water;

whence: $F_2 = z . a_2 b_2$

H_1 = the mean height of the points of entrance of the water above the lower water surface.

H_2 = the mean height of the points of discharge of the water above the lower water surface.

We thus have: $H_1 - H_2$ = the positive or negative height through which the water flows up or down in the wheel.

Relatively to the several velocities, let:

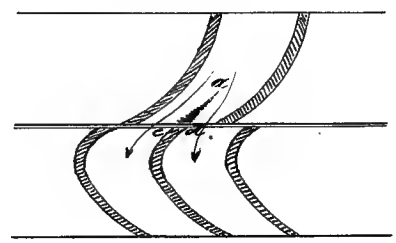
u = the absolute velocity of the water as it enters the wheel;

v = the absolute velocity of the wheel;

w = " " relative " " " " water to the wheel.

These velocities are to be understood as the mean normal velocities or those which exist under the normal relations and circumstances.

(Fig. 62.



As the water enters the moving-wheel with the absolute velocity (u), the latter is combined with a contraction of the jet due to the thickness of the buckets, and hence also with an increase of velocity over the initial velocity (u). If we conceive the buckets in the guide- and moving-wheel to be in a position such as is shown in Fig. 62, the thickness (cd) of the bucket in the moving wheel presents a resisting surface to the free flow of the water through the canal in the guide wheel; and a sort of wedge (acd) of water will be formed, to either side of which the water flows into the bucket spaces. The area of the orifice is however diminished by the area of base of this wedge, or the area of the edge of the bucket, and the velocity of discharge of the water from the guide wheel into the moving wheel will consequently be increased. Accordingly let:

u_1 = the absolute velocity of the water immediately after its entrance into the moving wheel;

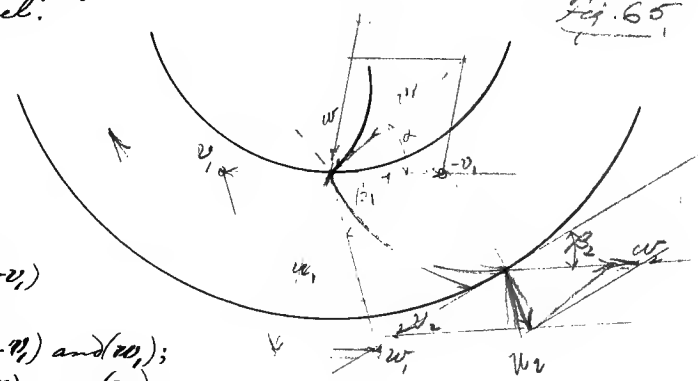
u_2 = the absolute velocity with which the water leaves the moving wheel after it has done its work;

v_1 = the mean velocity of the periphery of the moving wheel at the point of entrance of the water;

v_2 = the mean periphery velocity of the moving wheel at the point of discharge of the water;

We thus have: w = the resultant of (v) and $-(v_1)$;
Further: w_2 = the relative velocity of the water as it leaves the wheel; hence:
v_2 = the resultant of (v_2) and (w_2).
w_1 = the relative velocity of the water as it begins its motion thro' the moving wheel.

Fig. 65



In Fig. 65 is represented the resolution and composition of the several velocities in an inner-shot turbine
In this we have:
w = the resultant of $(-v_1)$ and (w);
v_1 = the resultant of $(+u_1)$ and (w_1);
v_2 = -"- -"- -"- (u_2) -"- (w_2).
These resultants have the same components also in an outer-shot and side-shot turbine.
Further let:
α = the angle formed by the directions of (v) and $-(v_1) = \hat{vv_1}$;
β_1 = -"- -"- -"- -"- -"- -"- $-(w_1)$ -"- $-(u_1) = 180° - \hat{v_1 w_1}$;
β_2 = -"- -"- -"- -"- -"- -"- -"- $-(w_2)$ -"- $-(u_2) = 180° - \hat{v_2 w_2}$;

h = the hydraulic head which causes the velocity (w);
h_1 = -"- -"- -"- -"- - - (w_1);
h_2 = -"- -"- -"- - - - - (w_2);
these heads furthermore denote the difference between the hydrostatic and atmospheric pressures at the respective points where these velocities occur.
Again let:
n = the number of revolutions of the turbine per second of time;
ω = -"- angular velocity;
$\pi = 3.1416$
$\omega = \frac{v_1}{r_1} = \frac{v_2}{r_2}$

The motion of the water from the upper to the lower water surface may be divided into 4 distinct motions: viz:-
(1°):- the descent or motion of the water from the upper water surface to the play space between the guide and moving wheels;
(2°):- the entrance of the water into the moving-wheel;
(3°):- its passage through the moving wheel; and:
(4°):- its motion from the points of discharge in the moving wheel to the water surface in the outlet or tail race

Relative to the motion of the water from the upper to the lower water surface, this as mentioned above may be divided into 2 parts.

In to establish the required mechanical conditions for each of these 2 parts it must be borne in mind to a definite known proposition in Hydraulics.

When the water moves in an inclined canal, then in a state of permanency (beharrende zustand) the sum of the height of head (druckhöhe) and height due to velocity for any cross section (F) is equal to the analogous sum of the head and height due to velocity for any cross section (F_1) beyond the first, plus the height of F_1 above F, minus the height due to the resistances (widerstände, Höhe) occurring during the motion of the water from F_1 to F. In this we have assumed that on all points of a cross section, a similar motion, and particularly also the same velocity exists; and we must understand by the height of F_1 above F the height of the centre of gravity of F_1 above the centre of gravity of F.

In this proposition it is furthermore assumed that the canal itself is in a state of rest; but if the canal has itself an absolute motion, then we must understand in the height due to velocity in the proposition the relative height due to velocity, and another member, viz. the supplementary force (ergänzungs kraft) for a rotary motion must then be added to the right side of the equation.

For the cross section F the sum of the

Equal to the analogous sum for F_1 plus the height of F_1 above F plus the work of the supplementary force of the motion for 1 kilogr. water for the motion from F_1 to F, minus the work which must be performed upon this water.

Let us consider first:- the motion of the water from the upper water surface to the play space between the guide and moving wheel. If h and w of the water at this play space are denoted by (h) and its velocity at that point by (w); hence we have:

$h + \frac{w^2}{2g}$ for the sum of the hydraulic head and the height due to the velocity (w) at this place; this sum must be equal to the analogous sum for the upper water surface minus the height due to the resistances at the place.

At the upper water surface the hydraulic head is $= 0$ = zero, and the height due to velocity also approximately $= 0$; there there remains only the height from the upper water surface to the play space, which is $= H - H_1$.

Let the value of the height due to the resistances in the motion from the upper water surface to the play space, be $= \varrho_0 H$; whence there follows the equation:

$$h + \frac{w^2}{2g} = H - H_1 - \varrho_0 H$$

(2°):- The motion of the water at its entrance on to the moving wheel.

We have here already to do with a relative motion, since the inside of the moving wheels rotate about the wheel axis.

. The hydraulic head has been denoted by (h_1) and the relative velocity with which the water leaves its motion in the channel by (w_1); hence

$$h + \frac{w_1^2}{2g} = \text{the sum of hydraulic head and height due to velocity}$$

. This sum must be equal to the corresponding sum for the cross section immediately before the entrance of the water in the wheel, plus the difference of height between the two cross sections, — minus the height due to the resistances encountered in this passage from the guide into the moving wheel. Here the difference $(h_0 - h)$ is infinitesimal small and may therefore be neglected like the first supplementary force.

The height due to the resistance on the latter canal, which is $= \varrho_1 H$, must nevertheless be taken into consideration, notwithstanding the short space in which it occurs, because the impact of the water against the inlet is during its entrance is not inconsiderable.

. Hence there follows the equation for the second section of the water:

$$h_1 + \frac{w_1^2}{2g} = h + \frac{w^2}{2g} - \varrho_1 H.$$

(3):— The motion of the water through the moving wheel.

This motion is also essentially a relative motion. For the terminal cross-section of the moving wheel, the hydraulic head was denoted by (h_2), the relative velocity of the water to the wheel by (w_2); hence we have for the sum of head and height due to velocity the expression:

$$= h_2 + \frac{w_2^2}{2g}$$

This sum must be equal to the hydraulic head (h) for the preceding cross section, plus the work of the first subsupplementary force of the relative motion, which is here the centrifugal force.

. For if (ω) is the angular velocity of the moving wheel, then the centrifugal force at a distance $= r$ from the axis is $= \frac{\omega^2}{g} \cdot r$.

If (r) varies by the differential (dr), then the work performed by one kilogramme of water will be: $= \frac{\omega^2}{g} \cdot r \cdot dr$

. But since this distance varies from (r_1) to (r_2), so that the total work of the centrifugal force becomes:

$$= \frac{\omega^2}{g} \int_{r_1}^{r_2} r \cdot dr = \frac{\omega^2}{g} \cdot \frac{r_2^2 - r_1^2}{2};$$ and furthermore because:

$$\omega \cdot r_1 = v_1,$$

and: $\omega \cdot r_2 = v_2$, this work also becomes:

$$= \frac{v_2^2 - v_1^2}{2}.$$

From the form of this equation it is evident that this work may become either positive, negative, or even $= 0 =$ zero.

In addition there must be added to this expression the difference between the heads at the initial and terminal cross section, viz: $H_1 - H_2$; and there must also be deducted the height due to the resistance encountered on the motion through the moving wheel; let this latter be: $= \varrho_2 H$; hence there follows the equation:

$$h_2 + \frac{w_2^2}{2g} = h + \frac{w_1^2}{2g} + H_1 - H_2 + \frac{v_2^2 - v_1^2}{2g} - \varrho_2 H.$$

The height due to the velocity (w_1) at the initial cross section is: $= \frac{w_1^2}{2g}$.

(A°):— The motion of the water from the beside of the discharge from the Moving wheel to the lower water surface:

We may here assume that the water has arrived at a state of rest at the lower water surface, and that the available velocity with which the water has left the turbine has been destroyed in this manner.

Hence the unknown and we call the height due to velocity, for the lower water surface $= c = \text{zero}$.

This O must be equal to the hydraulic head for the cross section at the point of discharge plus the height due to velocity of the latter, plus the difference in height between the this place of discharge and the lower water surface, viz: $= H_2 - o$, minus the height due to the resistance in this motion, viz $= \zeta_3 . H$.

Hence we have the equation:

$$O = h_2 + \frac{v_2^2}{2g} + H_2 - \zeta_3 . H .$$

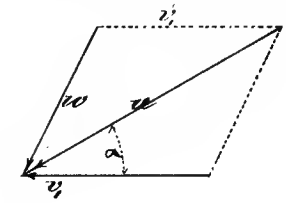
If we now add together these four equations for the separate motions, several members or terms may be eliminated, viz: all the Überwinde (= Höhen) hydraulische).

If we then unite all the heights due to velocity together on the left member or side of the equation, we shall then obtain finally:

$$\frac{u^2 - w^2 + w_2^2 - u_2^2 + v_1^2 - v_2^2}{2g} = H\left(1 - \zeta_0 - \zeta_1 - \zeta_2 - \zeta_3\right) = \varepsilon . H , \quad \ldots \quad (A)$$

where (ε) denotes the hydraulische Wirkungsgrad.

The left member of this equation now is transformed. If we consider the parallelogram of velocities at the entrance into the moving wheel, we have here.



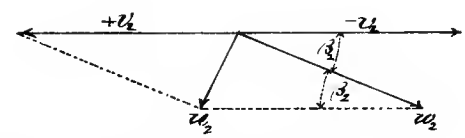
$$w^2 = u^2 + v_1^2 - 2u . v_1 . \cos . \alpha .$$

Considering next the parallelogram of velocities at the point of discharge of the water from the moving wheel, we shall obtain:

$$w_2^2 = v_2^2 + u_2^2 - 2 v_2 . u_2 . \cos . \beta_2$$

We thus have:

$$w^2 - w^2 + v_1^2 = 2 . u . v_1 . \cos . \alpha ;$$



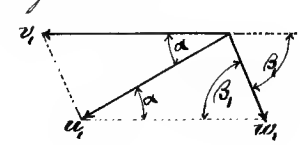
and: $$w_2^2 - u_2^2 - v_2^2 = 2 . v_2 u_2 \cos . \beta_2 - 2 v_2^2 ,$$

whence:

$$u^2 - u^2 + w_2^2 - u_2^2 + v_1^2 - v_2^2 = 2\left(u v_1 \cos . \alpha + v_2 u_2 \cos . \beta_2 - v_2^2\right) .$$

Substituting this value in the above equation (A) there follows:

$$\varepsilon = \frac{u v_1 \cos . \alpha + v_2 u_2 . \cos . \beta_2 - v_2^2}{g H} ,$$



or:

$$\varepsilon = \frac{v_2^2}{gH}\left[\frac{u}{v_2} . \frac{v_1}{v_2} . \cos . \alpha + \frac{u_2}{v_2} . \cos . \beta_2 - 1\right] ,$$

in which we may furthermore place: $\frac{v_1}{v_2} = \frac{r_1}{r_2}$

This formula might be used to establish a general formula by which the conditions, for which the working coefficient (η) of a turbine becomes a maximum, would be indicated.

For example, if the dimensions and hence also the Turbine itself is given; and also the fall H, the question may arise, what would be the most advantageous run of the wheel, and hence what values of (v_1) and (v_2) are most advantageous. These values are manifestly those for which the working coefficient (η) of the Turbine becomes a maximum.

The working coefficient (η) may now be determined by means of the hydraulic coefficient (ε); since we have:

$$\eta.\varepsilon = \varepsilon.\varphi - \mu$$

In this equation the coefficient (φ) is the same as was used in determining the relative loss of water, which is $Q(1-\varphi)$, and (μ) the coefficient of loss produced by the axial friction. This last value of coefficient (μ) we will assume to be constant, and will regard only the hydraulic working coefficient (ε), so that the most advantageous value of (η) shall be determined by the most advantageous value of (ε), and hence that $\eta)$ shall become a maximum when (ε) is a maximum.

In order to find the most advantageous run $(\text{prop}\ \varepsilon\eta)$ from the above established formula for which the working effect (η) will become greatest, we should have to proceed in the following manner.

The sum of all these disposable forces of head, viz:

$$H\left(\varphi_0 + \varphi_1 + \varphi_2 + \varphi_3\right)$$

may be expressed by means of the familiar empirical hydraulic formulæ, by means of a certain coefficient of resistance, by means of certain cross sectional dimensions, and by means of the velocity of the flowing water at various points.

All these velocities may furthermore be reduced to a definite velocity, as for instance to (v_2) with which the water flows out, so that we may finally place:

$$H\left(\varphi_0 + \varphi_1 + \varphi_2 + \varphi_3\right) = H(1-\varepsilon) = A.\frac{v_2^2}{2g}$$

From this we obtain:

$$\varepsilon = 1 - A\frac{v_2^2}{2gH} = 1 - A.\frac{v_2^2}{2gH}\left(\frac{w_2}{v_2}\right)^2$$

We have however another expression for (ε). In this expression we have the two angles (α) and (β_2) which are to be considered as given, when the Turbine and $\frac{r}{r_2}$ and $\frac{v_1}{v_2}$ are given. The velocity (w) may also be here reduced to (w_2), so that we may write:

$$\varepsilon = \frac{v_2^2}{2.g.H}\left[B.\frac{w_2}{v_2} - 1\right],$$

where $B)$ is a value analogous to (A), which is known, when the Turbine is given.

From the combination of these two expressions for (ε), we may arrive at an expression for the most advantageous values of (v_2) and (w_2), and hence for the most advantageous run of the Turbine. If we place:

$$v_2^2 = \lambda.2.g.H,$$

the expressions for (ε) may then be written as follows.

$$\varepsilon = 1 - A.\lambda\left(\frac{w_2}{v_2}\right)^2 = 2.\lambda\left(B.\frac{w_2}{v_2} - 1\right),$$

whence there follows: $\frac{w_2}{v_2} = f(\lambda)$

Substituting this expression for $\frac{w_2}{v_2}$ as a function of (λ) and their known magnitudes in the second equation for (ε), we shall obtain:

$$\varepsilon = 2\lambda\left(B.f(\lambda) - 1\right);$$

and we may now ask for which value of (λ) this expression becomes a maximum.

Assuming this value of (λ) to be $=(\lambda')$, we shall then have for the most advantageous velocity (v_2):

$$v_2' = \sqrt{\lambda'\ 2.gH},$$

whence there results the maximum value of (ε) which corresponds to the most advantageous run:

$$\text{max.}\varepsilon = 2\lambda'\left(Bf(\lambda') - 1\right),$$

and: $$\text{max.}\eta = q.\text{max.}\varepsilon - \mu.$$

The magnitudes (A) and (B) are only evident of (H), but are functions of certain dimensions, angles, and coefficients of resistance. Accordingly (λ') is also independent of (H), and hence:

the most advantageous velocity is proportional to $\sqrt{H}$

Accordingly if the same turbine is used for different falls (H), the most advantageous run of the wheel will vary for the different falls.

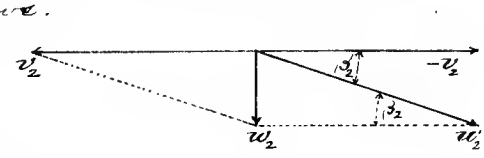
In this it is assumed that the various resistances of the cross-sections which have ~~to be~~ must become ~~overcome~~ do not change with the fall (H) which will in general be the case only when all the cross-sections are completely filled with the flowing water.

From this we may readily see that the rule in a certain measure strict pursuit of the mode of procedure indicated above is connected with great complications and extensions which will finally give only a not very accurate expression for the maximum value.

A shorter mode of procedure, with regard to practical necessities may therefore be adopted, in which we start from the following two assumptions. We may assume with great probability:—

(1°):— That it is very important in order to secure the most advantageous run of the turbine, that the water shall enter entirely without impact into the moving wheel

(2°): That it is very advantageous when the water flows off from the turbine in such a direction that the absolute velocity of discharge shall be directed normally to the periphery velocity of the wheel, in order that the water may be removed as quickly as possible from the turbine, and offer no resistance to the water which follows.



The second assumption may be reduced to the following:

Let (w_2) be directed normally to (v_2). The three velocities $(w_2)(v_2)$ and (w_2) will thus form a right angled triangle, in which the velocity (w_2) is the hypotenuse, and (w_2) opposite to the angle (β_2).

Hence we also have: $v_2 = w_2 \cdot \cos.\beta_2$,
which substituted in the formula for (2) page —

$$E.g.H. = u.v_1.\cos.\alpha.$$

If we had furthermore assumed, as a condition for the most favorable run of the turbine, that the water shall enter without impact, then this condition would be fulfilled provided that the direction of the relative velocity (w) with which the water enters the Turbine, shall coincide with the direction of the relative velocity (w_1) of the water upon its entrance; and the common direction of these two velocities must be tangential to the first element of the bucket's curvature.

This however is true only when we neglect the impact of the water against the edge (Stirnfläche) of the buckets, and the impact of the water particles among each other.

We have denoted the angle made by the direction of (u) with (v_1) by (α), and the supplement of the directions of (v_1) and (w_1) by (β_1). Now if (w) and (w_1) coincide, then (β_1) becomes equal to the angle opposite to the side (u) in the parallelogram of velocities; and hence we have:

$$\frac{u}{v_1} = \frac{\sin.\beta_1}{\sin.(\alpha+\beta_1)}.$$

Substituting this value of the ratio in (what?) — gives for us

$$E.g.H = v_2^2 \frac{\cos.\alpha \cdot \sin.\beta_1}{\sin.(\alpha+\beta_1)},$$

whence:

$$v_1 = \sqrt{\varepsilon} \cdot \sqrt{g.H. \frac{\sin.(\alpha+\beta_1)}{\cos.\alpha.\sin.\beta_1}}$$

The general form of this formula appears to be useful for practice; although the coefficient $\sqrt{\varepsilon}$ does not agree with the former general formula. It appears that this coefficient to which the two angles must be multiplied is $< \sqrt{\varepsilon}$

Hence there appears to be something defective in the above assumptions.

The first assumption may not be altered. The second assumption accordingly appears not to be exactly true; nor is this strange, since the impact which occurs during the entrance of the water into the wheel is not only an impact against the bucket itself, but also an impact against the edge of the bucket, as well as between the water particles themselves.

But in order to obtain the most simple formulae, we will make another analogous assumption, viz: that it conduces to the most advantageous run of the Turbine when the directions of the velocities (w_1) and (w) coincide, in which $w_1 < w$.

The question then arises whether this assumption will also lead to an analogous formula in which the quotient for (ε) will be smaller.

We have: $\frac{u_1}{v_1} = \frac{\sin.\beta_1}{\sin.(\alpha+\beta_1)}$,

and if we place:

$\frac{u_1}{w} = \frac{\varepsilon - \lambda}{\varepsilon}$,

we shall obtain:

$$\varepsilon.g.H = \frac{\varepsilon}{\varepsilon-\lambda}.u_1 v_1.\cos.\alpha = \frac{\varepsilon}{\varepsilon-\lambda} v_1^2 \frac{\cos.\alpha.\sin.\beta_1}{\sin.(\alpha+\beta_1)} .$$

From this there would follow for the most advantageous velocity:

$$v_1 = \sqrt{\varepsilon-\lambda}\cdot\sqrt{g.H.\frac{\sin.(\alpha+\beta_1)}{\cos.\alpha.\sin.\beta_1}} .$$

This formula accords very well with the general formula; and in this, the coefficient of the expression:

$\sqrt{g.H.\frac{\sin.(\alpha+\beta_1)}{\cos.\alpha.\sin.\beta_1}}$ has also a smaller value.

For different Turbines (λ) will have different values, which may be determined separately for each kind of Turbine.

From this formula it would appear that the most advantageous run of the Turbine would be secured if the water would exert a certain impact or percussion against the rear surface of the bucket.

We have thus found for the most advantageous run of the Turbine the following formulas:

$$u_1 : v_1 : w_1 = \sin.\beta_1 : \sin.(\alpha+\beta_1) : \sin.\alpha \quad \text{---------} (1).$$

These are two independent equations.

$$u_2 : v_2 : w_2 = \sin.\beta_2 : \cos.\beta_2 : 1 \quad \text{---------} (2).$$

which are likewise two independent equations.

$$\varepsilon.g.H = u.v_1.\cos.\alpha , \quad \text{---------} (3)$$

whence by combination with equation (1):

$$v_1 = \sqrt{\varepsilon-\lambda}\cdot\sqrt{g.H\frac{\sin.(\alpha+\beta_1)}{\cos.\alpha.\sin.\beta_1}} , \quad \text{---------} (4)$$

in which:

$$\frac{\varepsilon-\lambda}{\varepsilon} = \frac{u_1}{u} .$$

From this we do not care to express (ε) theoretically; and hence we cannot determine the most advantageous run of the Turbine in a strictly mathematical manner.

In the above formulas we must introduce for different Turbines different coefficients determined by experiment; and we must now learn how these empirical mean values are found.

We accordingly put the question, how the most advantageous values of (ε) and (λ) are found for a given Turbine. This is done by means of **Prony's Brake**, ------ with which a series of experiments are made upon the Turbine, and from the resulting series of coefficients, the mean value is used.

At first the cheeks of the brake are pressed only a little against the Turbine, and we wait until the wheel has arrived at a state of permanency (Beharrungszustand). By means of a counting apparatus

the number of revolutions per minute are then measured, by another apparatus the moving water (aufschlagwasser) is measured, as well as the total fall (H), for which this state of permanency occure.

From the magnitudes (Q) and (H) we may then determine the absolute effect:

$$E_0 = 1000 . Q . H ,$$

which work is wholly consumed or taken up by the brakes if we leave the axle friction of the Turbine out of consideration.

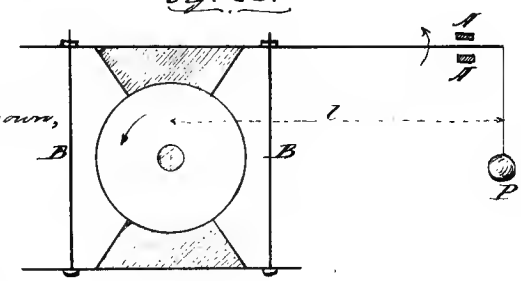
Denoting the intensity of the friction generated by the brakes on the circumference of the disc which is fastined to the axle of the turbine by (R), we then have for the total friction:

$$= R . 2\pi . r ,$$ where (r) is the radius of the disc.

Now if (n) denotes the number of revolutions per minute, of the wheel, we shall then have for the work destroyed or taken up by the friction per second: $E = \frac{n}{60} . R . 2\pi . r = 0{,}105 . n . R . r .$

This friction (R) must of course be measured, which is done by loading the brake lever at its extremity until it balances between the two points of resistance A, A, Fig. 70.

Fig. 70.



In this state we then have.

$$P . l = R . r ,$$

and: $$E = 0{,}105 . n . P . l .$$

For this formula, $(n, P$ and $l)$ are known, whence we may easily find:

$$\eta = \frac{E}{E_0} .$$

A new experiment is now made by increasing the pressure on the brakes, by means of the bolts B, B. In consequence of this increased pressure, (n) will become smaller and (P) greater; the lever will then strike against one of the points of resistance A, and the weight P must therefore be increased until the lever again balances freely between the points $A . A$. A new state of permanency, (beharrungs zustand) has then set in, and a new values of (E), and hence also one for (η) may be determined as above.

In this manner a series of values of (η) are found, as also a series of values of (n). We shall thus find that in the beginning as (n) decreases, (η) will increase, but that when (n) is further diminished, (η) will also becomes less.

From this series of corresponding values of (n) and (η), we may then readily find for which value of (n) the value of (η) will become a maximum.

Conceive this to be done, and let (n) be the corresponding number of revolutions per minute by which the greatest value of (η) results We thus have the maximum value of (ε), since:

$$\eta = \varphi . \varepsilon - \mu ,$$

whence: $$\varepsilon = \frac{\eta + \mu}{\varphi} .$$

In this last formula, we may place approximately:

$$\varphi = 1, \quad \text{and} \quad \mu = 0{,}03 \text{ to } 0{,}04$$

From the most favorable value of (n), we may now also determine the best velocity (v_1) of the turbine at the point where the water enters the moving wheel:

$$v_1 = \frac{n}{60} \cdot 2\pi \cdot r_1 = 0{,}105 \cdot n \cdot r_1 .$$

On the other hand, with the known values of (H), (α) and (β_1), we may also find the value of the expression:

$$v_1 = \sqrt{\varepsilon - \lambda} \cdot \sqrt{g H \cdot \frac{\sin.(\alpha+\beta_1)}{\cos.\alpha \cdot \sin.\beta_1}} = \sqrt{\varepsilon - \lambda} \cdot V_1 ,$$

whence:

$$\left(\frac{v_1}{V_1}\right)^2 = \varepsilon - \lambda ,$$

and:

$$\lambda = \varepsilon - \left(\frac{v_1}{V_1}\right)^2 .$$

In this manner the values of (ε) and (λ) may be found for various kinds of Turbines.

The coefficient (φ) may be calculated theoretically with some accuracy. It has the signification that:

$(1-\varphi)Q$ is the loss of water in cubic metres per second,

where (Q) denotes the quantity of water - " - - " - - " - - " - - " -
which constitutes the useful moving water (Nutzschlag).

This loss occurs chiefly in the play space between the guide and moving wheel, and we may therefore place:

$$(1-\varphi)Q = \mu' F' \sqrt{2g \cdot (h - h')} ,$$

where (F') denotes the area of the surface thro' which the discharge occurs (durchflussfläche); (μ') a certain coefficient of discharge, and (h') the hydraulic [illegible] in this space or surface and (h) - - - .

The coefficient (μ') must be determined experimentally.

The hydraulic [illegible] (h) [illegible] determined from the first of the 4 fundamental equations from whose combination the first expression for the hydraulic working coefficient (ε) was deduced. We have:

$$h + \frac{w^2}{2g} = 0 + H - H_1 - \varrho_0 H .$$

If we place the height due to the velocity (w) proportional to (H), i.e.:

$$\frac{w^2}{2g} = m H ,$$

there follows:

$$h = H(1 - \varrho_0 - m) - H_1 .$$

Relatively to the head (h'), which exists in the space where the loss of water in the turbine occurs, this may happen in various cases but particularly in the following three

(1°):- When the turbine is submerged, that is when the water passes from the guide into the moving apparatus under water, then (h') is equal to the height of the lower water surface above the mean point of entrance (b), Fig. 71.

(2°):- In a tube Turbine, the hydraulic [illegible] (h') may also be placed equal to the height of the place where the water passes from the guide into the moving wheel above the lower water surface,

that is, we may place: $h' = -H_1$, and hence, also:

$$h - h' = H(1 - \varrho_0 - m).$$

(3°):- When finally the Turbine moves in the air, then (h') becomes $= 0$ = zero, since the exterior pressure at the point where the water enters the moving wheel is equal to the atmospheric pressure, and hence the hydraulic [illegible] $k = 0$.

But in order to obtain a uniform expression for $(h - h')$ for this case also, we will assume that the height (H_1) or loss is included in the loss of head $(\varrho_0 H)$. At all events the value of (H_1) is here very small, being the same as the amount of suspension and thus representing a loss of head. We may therefore place in all cases:

$$(1 - \varphi) Q = \mu' F' \sqrt{2gH(1 - \varrho_0 - m)}.$$

From this formula we may readily perceive that the turbine should be constructed in such a manner that the proportional number (m) shall as nearly as possible become $= (1 - \varrho_0)$. Were (m) greater than $(1 - \varrho_0)$, then water or air might be sucked into the wheel from the outside, thereby causing irregularities in the action of the turbine. This factor (m) is thus an important element in the construction of these wheels. Suppose that (m) is equal to or a very little less than $(1 - \varrho_0)$. When $m = 1 - \varrho_0$, we should then have:

$$\frac{w^2}{2g} = (1 - \varrho_0) H;$$

the entire capacity for work (arbeitsvermögen) corresponding to the disposable head or fall, after deducting that portion which is lost by the hydraulic resistances, is thus stored up in the water in the form of living force, at the place where the water passes from the guide into the moving wheel.

Fig. 71.

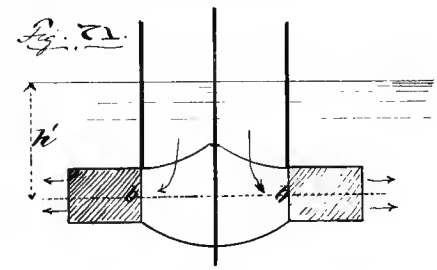
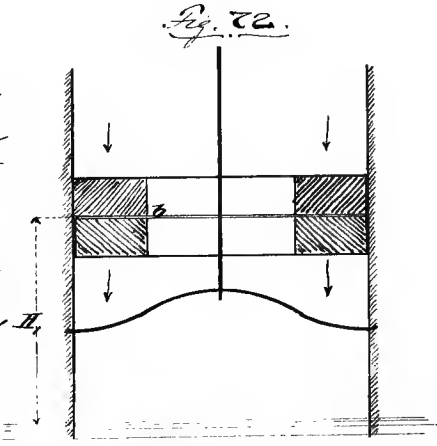


Fig. 72.



When the above equation is fulfilled, we thus perceive that the water acts solely by its living force which it has acquired while yet exterior to the apparatus; nor does this mode of action stand into any constant relation to the water in the feeder (Zuflusswasser).

Such a turbine may therefore be used according to circumstances either as a full or partial turbine, without the working coefficient being altered, since the entire capacity for work is held in the water in the form of living force.

But with this advantage a disadvantage is likewise combined. Since the water in such a case enters the moving apparatus without the least [illegible] it is not compelled to fill out completely the various cross sections of the canals in the moving wheel, and hence empty spaces may occur in these canals. Accordingly when such a turbine whose canals are

not completely filled by the flowing water is submerged in the lower water, the latter may then set back into these canals and fill out the various empty spaces or vacuums with a certain volume of dead water. Such a turbine may therefore run very well in air, but not when submerged.

On the other hand, when a turbine is so constructed that (m) is consid=erably less than $(1-\varrho_0)$, the relations are entirely different. In this case only a small portion of the total working capacity is held by the water as living force, while by far the greater portion of the water acts by pressure against the surface of the buckets. This state of course requires a constant relation with the water in the feeder, or upper water. Such a turbine therefore suffers a very noticable diminution of its working coefficient as soon as it becomes only partially supplied. (Brauschlagt wird) It has however this advantage that it may move either in the air or under water without its working coefficient being sensibly diminished.

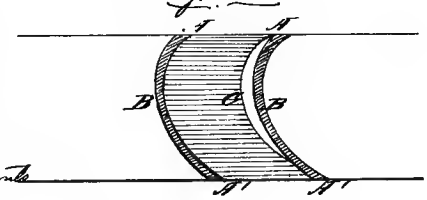
For the first class of turbines $(m) = 1-\varrho_0$, and these wheels are therefore called *Action Turbines*.

The second class, in which $(m) < 1-\varrho_0$, are called *Reaction Turbines*.

In order to combine the advantages of Action and Reaction Turbines, the wheel may be constructed in principle as an Action Turbine, but with the advantage of the reaction wheel, viz: its not sucking in water into the hollow spaces.

Assuming therefore that the Turbine is constructed essentially as an Action Turbine, the relations may be so chosen that the initial and terminal cross sections of the canals in the moving wheel shall be completely filled with the flowing water.

Fig. 73.



This however will then not generally be the case for the intermediate cross section BB, Fig. 73, which will not be completely filled out. This empty space may be filled by means of "back-buckets" ACA', which amount to the same as if the thickness of the buckets was increased at these intermediate points, or as if the thickness was variable. If the law of this variation in thickness has been chosen correctly, then the turbine is an action wheel, without however, the defect of sucking water into empty spaces.

Partial Turbines must thus under all circumstances be construc=ted as Action Turbines. This class of turbines is generally used when the head or fall is quite considerable, and where the dimensions for a full turbine would become too small. In this case also a portion of the total head may be used in order to permit the tur=bine to move in the air, thus obviating entirely the danger of the back-water.

From the above Consideration it is readily seen that the coefficient (m) plays an important part with reference to the manner in which the water acts in the Turbine; and it therefore deserves to be considered as a special element of construction.

The value of (m) is thus the characteristic of an Action or a Reaction Turbine; and we will accordingly write further as principal equations:

$$\frac{v_1^2}{2g} = m.H, \qquad \text{(5)}.$$

and:

$$\frac{r_1}{r_2} = \frac{v_1}{v_2} \qquad \text{(6)}.$$

These six equations represent (8) independent relations between the given or approximately found values: (H, ε, and λ) and the following 12 construction elements, viz:

$$m;\ \alpha,\ \beta_1,\ \beta_2;\ \frac{r_1}{r_2},\ w,\ w_1,\ w_2;\ v_1,\ v_2;\ u_1 \text{ and } u_2.$$

Of these elements, we may thus assume the values of (4) and determine the rest by means of the above equations

These Equations may be combined in many different ways; but of all possible combinations, the following are worthy of notice.

(1): The combination of (3) and (4).

Developing the value of (v_1) from eq. (3) we have:

$$v_1 = \frac{\varepsilon . g.H}{u.\cos.\alpha} = \frac{\varepsilon}{\cos.\alpha} \cdot \frac{g.H}{\sqrt{2m.g.H}} = \frac{\varepsilon}{\cos.\alpha}\sqrt{\frac{g.H}{2.m}} \qquad \text{(a)}$$

in which the value of (u) is taken from eq. (5). This value of (v_1) is thus the most favorable periphery velocity at the points of entrance of the water.

Squaring this last value of (v_1), and also eq. (4), and combining the results, we shall obtain:

$$\frac{\varepsilon^2}{\cos^2\alpha} \cdot \frac{g.H}{2m} = (\varepsilon - \lambda) g.H. \frac{\sin.(\alpha+\beta_1)}{\cos.\alpha . \sin.\beta_1}$$; and by dividing by (gH), there follows:

$$2.m.\frac{\varepsilon - \lambda}{\varepsilon^2} = \frac{\sin.\beta_1}{\cos.\alpha . \sin.(\alpha+\beta_1)} \qquad \text{(b)}.$$

In like manner the relation:

$$\frac{w_1}{w} = \frac{\varepsilon - \lambda}{\varepsilon}$$

is no new equation, but is found by the combination of equations (1, 3, and 4). Equations (a) and (b) are also only combinations, and not new determining equations.

In order to find other new relations between these given magnitudes and the other elements of the wheel, we will consider the case of a Full Turbine.

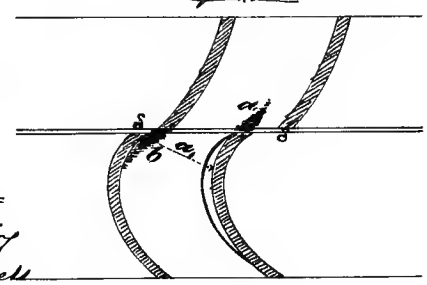
Full Turbine.

A full Turbine may also be used under certain circumstances as a partial Turbine, yet it was designed to be completely filled, and attains its maximum effect only on this condition. Under all circumstances, the characteristic ($\pi\pi$) is not to be made fully equal to $1-\varrho_0$, but its value should always be a little less, in order that a [illegible] may constantly exist at the place where the water passes from the guide into the moving apparatus. In this case the initial cross section of the canals in the moving wheel is sure to be completely filled out with the flowing water, as also the terminal cross section, although some intermediate cross sections may not be perfectly filled. When the terminal cross section is not filled out, it may be due to a greater relative velocity (w_2) than is required; since the greater the value of (w_2) the greater also that of (v_2) which, however, should always be as small as possible. It is therefore advisable to build the turbine in such a manner that the terminal cross section shall be completely filled.

The conditions that the initial and terminal cross sections of a canal in the moving wheel, and also that the terminal cross section of a canal in the guide apparatus, shall be completely filled lead to certain other equations.

On account of the thickness of the buckets, it is readily seen that the initial cross section of a moving-canal can not be perfectly filled with regularly flowing water, since before each edge of a bucket a wedge shaped body (a) or (b) Fig. 74, of water is formed which does not participate in the properly flowing motion of the water. Consequently the terminal cross-section of the guide canal and the initial cross section of the moving canal are diminished, the first by the thickness of the moving wheel buckets and the latter by the thickness of the guide buckets.

Fig. 74.



If this narrowing or diminution of cross section did not exist, we should then have for the quantity of water flowing out of the guide wheel:

$$Q = z_0 . a_0 . b_1 . w ;$$

but since the thickness of the buckets can not be neglected, this value of (Q) must accordingly be multiplied by a certain coefficient of discharge (k_0) which is less than (1) = unity; whence:

$$Q = k_0 . z_0 . a_0 . b_1 . w .$$

But a certain loss of water occurs through the play space at the spot where the water passes from the guide into the moving wheel; we thus have for the amount of water flowing into the moving wheel:

$$\varphi Q < Q ,$$

and:

$$\varphi Q = k_1 . z . a_1 . b_1 . w_1 ,$$

where (k_1) is a coefficient which must be used on account of the contraction

of the initial cross section of the canal in the moving wheel.

During the passage of the water thro' the moving apparatus no further loss of water takes places, and hence the amount of water which is discharged is:

$$\varphi Q = z . a_2 . b_2 . w_2 .$$

No coefficient (k) is required here because no contraction of the terminal cross section occurs.

It will easily be seen that the coefficients (k_0 and k_1) are purely empirical, and are found as follows without sensible error. The coefficient (k_0) depends upon the thickness of the buckets in the moving wheel, and (k_1) -"- -"- -"- -"- -"- -"- -"- -"- -"- guide -"-

The width of the initial cross section of a canal in the moving wheel was denoted by (a_1), and the thickness of the buckets by (δ_1); hence we may place:

$$k_0 = \frac{a_1}{a_1 + \delta_1}$$

and in like manner:

$$k_1 = \frac{a_0}{a_0 + \delta_0}$$

When back buckets are present, the thickness (δ_1) becomes considerably greater, in which case it may be taken directly from the drawing in the following manner.

Let ($a c t$) and ($a' c' t'$) be the middle lines of the canals in the moving and guide wheel; at (a') draw a tangent to this middle line ($a' t'$) and ($a' c'$) normal to this tangent. The line ($a' b'$) will then be $= \frac{1}{2} . a_0$, and $b' c' = \delta_0$. In like manner $ab = \frac{1}{2} . a_1$, and $bc = \delta_1$.

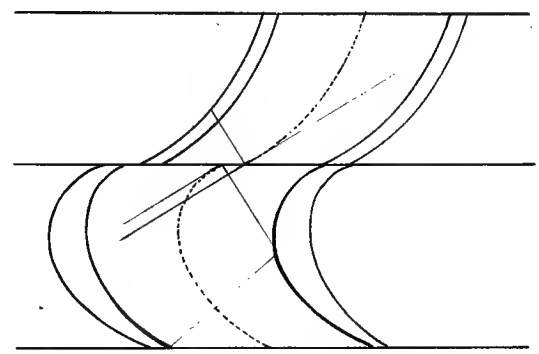
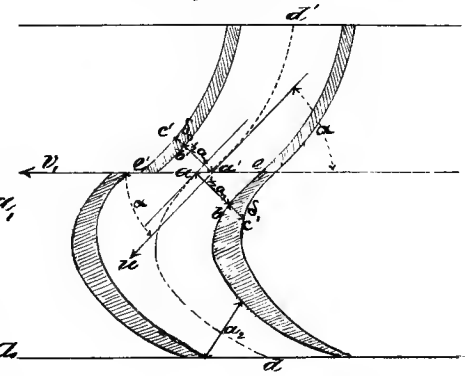


Fig. 65.



From the above three equations for (Q) and (φQ), the following three equations for (a_0, a_1 and a_2) may be deduced.

The mean circumference ($2\pi r_1$) of the guide wheel at the place where the water enters into the moving wheel is at all events equal to the mean circumference of the moving wheel at this place; and if this circumference is divided by the number of buckets (z_0) in the guide wheel, then

$cc' = \frac{2\pi r_1}{z_0}$ will be the length of the arc between the middle points of two consecutive buckets, which arc may be considered as a straight line without sensible error. But as (α) is the angle which the middle line (or its tangent at (a')) of the guide canal makes with the horizon, we accordingly have from the above definition:

$$a_0 = \frac{2\pi . r_1}{z_0} . \sin . \alpha - \delta_0 \; ; \quad \ldots \ldots \ldots \ldots \ldots \quad (1).$$

and in like manner:

$$a_1 = \frac{2\pi . r_1}{z} . \sin . \beta_1 - \delta_1 \; ; \quad \ldots \ldots \ldots \ldots \ldots \quad (2).$$

as also:

$$a_2 = \frac{2\pi . r_2}{z} . \sin \beta_2 - \delta_2 \; \ldots \ldots \ldots \ldots \ldots \quad (3)$$

These six equations are not new determining equations, but are only equations of condition for the following 11 other turbine elements which are not included in the 12 mentioned above:

$$a_0 \, , \; a_1 \, , \; a_2 \, ; \; \delta_0 \, , \delta_1 \, , \delta_2 \, ; \; r_1 \, , \; b_1 \, , b_2 \, ; \; z_0 \, , \text{ and } z \, .$$

(End of 1st team.).

The former equation (b): (p. 96)

$$2m . \frac{\varepsilon - \lambda}{\varepsilon^2} = \frac{\sin . \beta_1}{\cos . \alpha . \sin . (\alpha + \beta_1)} \, , \text{ may be written in the following}$$

form:

$$2m . \frac{\varepsilon - \lambda}{\varepsilon^2} . \frac{\cos . \alpha}{\sin . \beta_1} \left(\sin . \alpha . \cos \beta_1 + \cos . \alpha . \sin \beta_1 \right) = 1 \, ,$$

or:

$$\frac{\varepsilon - \lambda}{\varepsilon^2} . m . \sin . 2\alpha \left(\operatorname{Cotg} . \alpha + \operatorname{Cotg} . \beta_1 \right) = 1 \; \ldots \ldots \ldots \ldots \quad (b').$$

All of the above general equations are true for every kind of Turbine, but they were developed more especially for the case of a Full Turbine.

The terminal cross section of the guide canals will always be completely filled out with water, and we should therefore have for the quantity of water flowing out from the guide wheel if no obstructions presented themselves: $Q = z_0 . a_0 . b_1 . w$;
this is however not the case, since the apertures or terminal cross sections of these canals are narrowed by the thickness of the buckets in the guide wheel. This value must therefore be multiplied by a correction-coefficient (k_0) in order to obtain the true value for this water volume which flows out from the guide wheel.

[illegible]

$$Q = k . k_0 . z_0 . a_0 b_1 . w .$$

But from the leakage or waste thro' the play space, the quantity of water which actually flows to the moving wheel is less than the above quantity (Q); and we must thus multiply this last quantity (Q) by a coefficient of loss (φ) which must be less than (1) = unity.

Furthermore, as the area of a cross section of a canal in the moving-wheel when the buckets are conceived as having no thickness is:

$$z . a_1 . b_1 \, , = \text{the initial cross section} \, ;$$

we must multiply this area by a correction coefficient (k_1) when the thickness of the buckets is taken into consideration, which must always be done. This product: $k_1 . a_1 b_1 . z$ is thus the aliquot part of the total area in the moving wheel canals which is filled

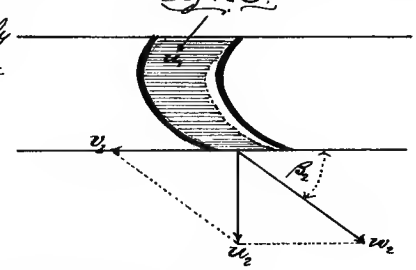
with regularly flowing water (regelrecht fliessendes wasser). Multiplying this aliquot area by the relative velocity (w_1) with which the water enters the canals in the moving wheel, we have for the quantity of water that enters per second into one of these canals:

$$\varphi Q = k_1 . z . a_1 . b_1 . w_1 = z . a_2 . b_2 . w_2 ,$$

which last expression is the quantity of water that leaves the moving wheel per second. This is precisely equal to the amount which enters, since no loss occurs during the passage of the water thro' the wheel nor are any obstructions whatever presented to its free discharge, and hence also a correction coefficient is not needed for the discharge thro' the terminal cross section.

When the terminal cross section is not completely filled out with regularly flowing water, it may be due to an improper angle (β_2) of discharge

Fig. 76.



The above mentioned 11 elements of the wheels' construction, viz:

$$a_0, a_1, a_2; \delta_0, \delta_1, \delta_2; r_1, b_1, b_2; z_0 \text{ and } z$$

are directly connected with the above (6) equations of condition, and if (5) of these elements are assumed, the remaining six may be determined.

For examining these (6) equations, we may [illegible] equations for some of the elements enclosed in the 12 mentioned on page 96.

$$m; \alpha, \beta, \delta_2; \frac{r_1}{r_2}, \ldots$$

From the three equations:

$$Q = k . k_0 . z_0 . a_0 . b_1 . w ; \quad \text{---------- (1).}$$

$$\varphi Q = k_1 . z . a_1 . b_1 . w_1 ; \quad \text{---------- (2).}$$

and:

$$\varphi Q = z . a_2 . b_2 . w_2 , \quad \text{---------- (3).}$$

we have, by combining (1) with (2), remembering that $\frac{w_1}{w} = \frac{\varepsilon - \lambda}{\varepsilon}$,

page 96:

$$\varphi = \frac{k_1}{k . k_0} . \frac{z}{z_0} . \frac{a_1}{a_0} . \frac{w_1}{w_1} . \frac{\varepsilon - \lambda}{\varepsilon} ; \quad \text{---------- (4)}$$

and by combining (1) with (3):

$$\varphi = \frac{1}{k_1} . \frac{k_1}{k . k_0} . \frac{z}{z_0} . \frac{a_2}{a_0} . \frac{b_2}{b_1} . \frac{w_2}{w_1} . \frac{\varepsilon - \lambda}{\varepsilon} . \quad \text{---------- (5)}$$

Let us now deduce the value of $\left(\frac{a_1}{a_0}\right)$ from the first value of (φ), and that of the ratio: $\left(\frac{a_2}{a_0}\right)$ from the second. For doing this we will first use an abbreviation for the cluster of coefficients by placing:

$$\psi = \varphi . k . \frac{k_0}{k_1} . \frac{\varepsilon}{\varepsilon - \lambda} ,$$

whence there follows:

$$\frac{a_1}{a_0} = \psi . \frac{z_0}{z} . \frac{w_1}{w_1} ,$$

and:

$$\frac{a_2}{a_0} = k_1 . \psi . \frac{z_0}{z} . \frac{b_1}{b_2} . \frac{w_1}{w_2} .$$

But from the first general equation, (eq. 1, page 91) we have:

$$\frac{w_1}{w_1} = \frac{\sin.\beta_1}{\sin.\alpha},$$

and for $\left(\frac{w_1}{w_2}\right)$ we may write:

$$\frac{w_1}{w_2} = \frac{w_1}{v_1}\cdot\frac{v_1}{v_2}\cdot\frac{v_2}{w_2},$$

which last becomes by substitution of the values of these ratios taken from general equations (1), (2) and (6), pages (91) and (96):

$$\frac{w_1}{w_2} = \frac{\sin.\beta_1}{\sin.(\alpha+\beta_1)}\cdot\frac{r_1}{r_2}\cdot\cos.\beta_2\ ;$$

Substituting these values in the above equations on the last page we have:

$$\frac{a_1}{a_0} = \psi\cdot\frac{3_0}{3}\cdot\frac{\sin.\beta_1}{\sin.\alpha}, \quad \text{---------- (6').}$$

and:

$$\frac{a_2}{a_0} = k_1\cdot\psi\cdot\frac{3_2}{3}\cdot\frac{b_1}{b_2}\cdot\frac{r_1}{r_2}\cdot\frac{\sin.\beta_1\cdot\cos.\beta_2}{\sin.(\alpha+\beta_1)}. \quad \text{---------- (7').}$$

These two equations may replace equations (2) and (3) on the preceding page. Combining equations (6') and (7'), thereby eliminating (a_0) we obtain:

$$\frac{a_1}{a_2} = \frac{1}{k_1}\cdot\frac{b_2 r_2}{b_1 r_1}\cdot\frac{\sin.(\alpha+\beta_1)}{\sin.\alpha\cdot\cos.\beta_2}.$$

Writing this last equation in another form, and multiplying by (k_1) we get:

$$i = k_1\cdot\frac{a_1}{a_2}\cdot\frac{r_2\cdot\sin.\beta_2}{r_1\cdot\sin.\beta_1} = \frac{b_2}{b_1}\left(\frac{r_2}{r_1}\right)^2\cdot\tang.\beta_2\cdot\frac{\sin.(\alpha+\beta_1)}{\sin.\alpha\cdot\sin.\beta_1} =$$

$$= \frac{b_2}{b_1}\left(\frac{r_2}{r_1}\right)^2\cdot\tang.\beta_2\cdot\frac{\sin.\alpha\cdot\cos.\beta_1+\cos.\alpha\cdot\sin.\beta_1}{\sin.\alpha\cdot\sin.\beta_1},$$

whence:

$$i.\cotg.\beta_2 = \frac{b_2}{b_1}\left(\frac{r_2}{r_1}\right)^2\cdot(\cotg.\alpha+\cotg.\beta_1) \quad \text{---------- (7)}$$

where (i) is used as an abbreviation merely for the above expression.

This last equation, (7), which contains 4 of the original 12 construction elements of the turbine, viz: $(r_1, r_2, \alpha, \beta_1, \beta_2)$ may be used as a <u>determining equation</u> for some of these 12 elements, and we will therefore number it (7).

If we place temporarily the value of the coefficient (i), $=1$, eq. (7) may then be used to determine several of the wheel elements, from which in turn we shall obtain a corrected value of the coefficient (i). by the subsequent substitution of the values so found in the expression for (i). That the value of this coefficient can not differ greatly from $1 =$ unity, will be seen from the following. Substituting in the expression for (i) the values of (a_1) and (a_2) taken from Eq's (2) and (3) page 99, we have:

$$i = k_1\cdot\frac{\frac{2\pi}{3}-\frac{\delta_1}{r_1\cdot\sin.\beta_1}}{\frac{2\pi}{3}-\frac{\delta_2}{r_2\cdot\sin.\beta_2}},$$

which expression can not be greatly different from unity $=1$ since the coefficient (k_1) is only a little less than $(1) =$ unity.

Bearing in mind the relation (b), page 99, equation (7) may be put into a more convenient form for many purposes.

Deducing the value of $(\text{tang}.\beta_2)$ from eq.(7), we obtain:

$$\text{tang}.\beta_2 = i.\frac{c_2}{a_2}\left(\frac{r_1}{r_2}\right)^2.\frac{\varepsilon-\lambda}{\varepsilon^2}.m.\sin.2\alpha \qquad \text{---} \quad (7.a).$$

Further by substituting the value of (ψ) in the expression for $(\frac{a_1}{a_0})$, eq.(6') on the foregoing page, we have:

$$\frac{a_1}{a_0} = \varphi.k.\frac{k_0}{k_1}.\frac{\varepsilon}{\varepsilon-\lambda}.\frac{z_0}{z}.\frac{\sin.\beta_1}{\sin.\alpha} = \varphi k.\frac{a_1}{a_1+\delta_1}.\frac{a_0+\delta_0}{a_0}.\frac{\varepsilon}{\varepsilon-\lambda}.\frac{z_0}{z}.\frac{\sin.\beta_1}{\sin.\alpha},$$

by substituting the values of (k_1) and (k_0) on page 98. Dividing thro' by $(\frac{a_1}{a_0})$, and introducing the values of (a_0) and (a_1) from eqs (1) and (2), page 99, we get:

$$1 = \varphi.k.\frac{\varepsilon}{\varepsilon-\lambda}.\frac{2\pi r_1.\sin.\alpha}{2\pi.r_1.\sin.\beta_1}.\frac{\sin.\beta_1}{\sin.\alpha} = \varphi.k.\frac{\varepsilon}{\varepsilon-\lambda},$$

whence we derive:

$$k = \frac{\varepsilon-\lambda}{\varphi.\varepsilon}.$$

For [illegible] value k can [illegible] differ from 1) = unity, and we must therefore assume that the relative velocity is increased [illegible] passage through the orifice.

When equations (7) or (7a) is considered as a determining equation for the original 12 elements, we then have for the remaining elements the following 4 relations, which have already been written:

$$Q = k.k_0.z_0.a_0.b_1.w;$$

$$a_0 = \frac{2\pi r_1}{z_0}\sin.\alpha - \delta_0;$$

$$a_1 = \frac{2\pi r_1}{z}.\sin.\beta_1 - \delta_1;$$

and:

$$a_2 = \frac{2\pi r_2}{z}.\sin.\beta_2 - \delta_2.$$

The remaining 5 elements, $(\delta_0, \delta_1, \delta_2, z$ and $z_0)$ are generally determined from empirical rules. If (z) or (z_0) are chosen too great, a considerable loss results from the friction of the water against the great surface which the buckets present, and on the other hand when the number of buckets is too small, there will be a similar loss in consequence of the irregular action of the water

Two more elements must furthermore be assumed, for one of which we will take the ratio $(\frac{c_2}{c_0})$, which we will place in general $= 1$, although this value may be greater or less than 1 = unity, as may be seen by examining eq(7). For the other element we will choose such conditions by which the value of the radius (r_1) will become fixed. This may be done in 3 different ways. (1°):- Let us fix upon the number of revolutions per minute (n) of the turbine; this number however is given by the equation:

$$n = 9,55.\frac{v_1}{r_1},$$

whence there follows:

$$r_1 = 9,55.\frac{v_1}{n}.$$

(2°):— In an inner-shot Turbine, the water flows to the moving wheel in a circular tube whose radius is $= r_1''$. If (c_1) is the velocity of the water in the tube or pipe, which velocity is known, we then have:

$$Q = \pi r_1''^2 . c_1 ,$$

whence: $$r_1'' = \sqrt{\frac{Q}{\pi . c_1}} .$$

(3°):— The value of (r_1'') may also be fixed by assuming a value for the ratio $\left(\frac{b_1}{r_1''}\right)$. For we have:

$$Q = k . k_0 . k_1 (a_0 + \delta_0) z_0 . b_1 . u ,$$

since from the equation on page 98, we have: $a_0 = k_1 (a_0 + \delta_0)$. Substituting in this the value of (a_0) on the preceding page, we have:

$$Q = k . k_0 . k_1 . 2\pi . \sin . \alpha . r_1''^2 \frac{b_1}{r_1''} . u ,$$

whence there follows:

$$r_1'' = \sqrt{\frac{Q}{k . k_0 . k_1 . 2\pi . \sin . \alpha . \frac{b_1}{r_1''} . u}} .$$

In this last equation, we ought properly to know the exact values of the coefficients $(k, k_0$ and $k_1)$ before an application of the formula can be made; but we may however content ourselves with an approximate value of (r_1'') obtained by using empirical values for these coefficients, since the radius (r_1'') is usually made a round number.

Application. — When we are called upon to design a Turbine, the problem is usually such that a wheel shall be constructed which, with a given head (H), shall develop a given number (N) of horse-powers. In order to solve this problem, it becomes necessary to know in the first place something about the working coefficient (η), as this coefficient can not be determined a priori as in the case of the ordinary vertical water wheels; and in like manner some supposition must be made relatively to the coefficient (λ)

Accordingly, if we assume some approximate empirical values for (η) and (λ), the hydraulic working coefficient (wirkungsgrad) (ε) may then be determined, since we have:

$$\eta = \varphi \varepsilon - \mu ,$$

whence: $$\varepsilon = \frac{\eta + \mu}{\varphi} .$$

The value of the coefficient (ε) is thus several per cent greater than that of (η); since the coefficient (μ) for the axle friction is always several per cent. In general we may place: $\mu = 0{,}0\pm$.

The coefficient (φ) expresses that aliquot part of the water volume, which produces useful effect or work; in a reaction Turbine, the value of (φ) may be $= 1$, or a very little less. Since there is no sensible loss [illegible] of the play space; whereas in a reaction Turbine the loss due to this play space is generally much greater. This loss however is [illegible] that given in the earlier case, we may place: $\varphi = 0{,}96$.

and we thus may place in general:

$$\varepsilon - \eta = 0{,}04 \text{ to } 0{,}08 \quad (\text{see } 1).$$

Having thus assumed preliminary values for the above coefficients, we may now find the quantity of water required to feed the turbine; for we have:

$$1000 . Q H = \frac{1}{\eta} . 75 . N = \text{the absolute effect},$$

whence there follows:

$$Q = \frac{0{,}075}{\eta} . \frac{N}{H} .$$

Of the above 12 elements: n_j: $(m, \alpha, \beta_1, \beta_2; \frac{r_1}{r_2}, u, u_1, u_2; v_1, v_2; w_1 \text{ and } w_2)$, 3 may now be arbitrarily assumed: Let us assume for instance:

$$\alpha, \beta_1, \text{ and the ratio } \frac{r_1}{r_2}.$$

For the case of a reaction Turbine, the assumption of these three elements is quite proper.

We may furthermore assume the ratio $(\frac{b_1}{b_2})$, which may in general though not always be placed $\frac{b_1}{b_2} = 1$.

If its value is not $= 1$, let us assume it to be such for the present, at least, and then find an approximate value for the angle (β_2)

~~In this purpose we will~~ [illegible] ~~the expression~~ $(V_1) = (1)$ page 101 and thereby determine a value for the angle (β_2) which we will call $[\beta_2]$ it is approximative.

Should this value of the angle (β_2) prove too great, the velocity of the discharge will also become too great, thus causing a diminution of the working coefficient (η); and we must therefore diminish the value of the ratio $(\frac{b_1}{b_2})$ by increasing the denominator (b_2).

The velocity (v_1) which will insure the most advantageous run of the Turbine may then be calculated from equation (A), page 91, since in this every factor is known.

The absolute velocity (u) may then also be found from eq. (3), page 91, and likewise the characteristic (m), from which we shall learn whether the turbine will become an Action or a Reaction wheel.

If the resulting value of the characteristic (m) is not such as was desired, — or in other words, if we have previously determined whether the wheel shall be an action or a reaction Turbine, — we must then make such correction of the assumed values as will cause (m) to have the required value.

The radius (r_1) of the Turbine at the place where the water enters the wheel may next be calculated. This may be found by fixing upon (1°): — the number of revolutions (n) per minute, whence we obtain:

$$r_1 = 9{,}55 \frac{v_1}{n};$$

or (2°): — if a certain velocity (c_1) with which the water shall flow to the wheel is assumed, the value of the radius is found from the formula on the preceding page:

$$r_1 = \sqrt{\frac{Q}{\pi . c_1}};$$

or: (3°): — if the value of the ratio $(\frac{b_1}{r_1})$ is assumed, (r_1) is found from the eq.: (see preceding page.)

$$r_1 = \sqrt{\frac{Q}{k . k_0 k_1 . 2\pi . \sin . \alpha . \frac{b_1}{r_1} . u}}$$

In this equation, a certain approximate value for the product $(k \cdot k_0 \cdot k_1)$ of the coefficients (k) must be assumed, which has been found by experiment. This value may in general be taken:

$$k \cdot k_0 \cdot k_1 = \frac{4}{5}.$$

For one or the other of these three modes, the value of the radius (r_1) is to be determined; and as the ratio $\left(\frac{r_1}{r_2}\right)$ was assumed in the outset, the value of the radius (r_2) at the point of discharge of the water will thus also become known.

We are now enabled to obtain some idea relatively to the size of the turbine, and thence determine from empirical rules the number of the buckets (z_0) and (z), as well as their thicknesses: $(\delta_0, \delta_1, \delta_2)$.

The dimensions (a_0, a_1, a_2) of the cross sections of the canals in the guide and moving wheels may next be calculated.

From the formulas on page 102, the values of $(a_0$ and $a_1)$ may be determined definitely, since no unknown quantities occur in the respective formulas. The value for (a_2) found from these equations is however somewhat indefinite, since the approximate value $[\beta_2]$ of the angle (β_2) was employed in the solution of the Equation.

Further, from the now known values of (a_0) and (δ_0), the value of the coefficient (k_0) can now be found from the equation on page 98, as also the value of the coefficient (k_0) from the values of (a_1) and (δ_1).

The value of the third coefficient (k) is determined from the equation on page 102: $\quad k = \frac{\varepsilon - \lambda}{\varphi \cdot \varepsilon}$.

The dimension (b_1) of the cross section of the canals at the point of discharge from the guide wheel and at the point of entrance to the moving wheel can now be found from the formula:

$$Q = k \cdot k_0 \cdot k_1 \cdot z_0 \cdot a_0 \cdot b_1 \cdot w;$$

and from the known value of (b_1), that of (b_2) is found, since the value of the ratio: $\left(\frac{r_1}{r_2}\right)$ was assumed in the beginning.

By means of these values, the true value of the expression or coefficient (i) may be determined, from which in turn we obtain the corrected value of the angle (β_2). The value of (i) is found from the equation:

$$i = k_1 \cdot \frac{a_1}{[a_2]} \cdot \frac{r_2 \cdot \sin.[\beta_2]}{r_1 \cdot \sin.\beta_1};$$

and the corrected value of (β_2) from:

$$\operatorname{tang.} \beta_2 = i \cdot \operatorname{tang.}[\beta_2].$$

With these corrected values of (i) and (β_2), a corrected value of the dimension (a_2) can now be found.

If this corrected value of (a_2) differs considerably from the approximate value $[a_2]$, then the foregoing operations must be repeated, and we shall thus obtain a new set of values for the above elements, by which the Turbine is completely determined

If it is required to design an action Turbine, the value of the characteristic (m) is fixed in the outset

In Reaction Turbines, on the other hand, the wheel elements are so dependent upon the value of (m), that in general it is advisable to allow the value of the characteristic to be determined by the results obtained by making other assumptions. Even in action Turbines, this plan is often the best.

If we assume the following values:

$$m, \quad \frac{r_1}{r_2}, \text{ and } \alpha,$$

then the value of the angle (β_1) results from the equation:

$$\operatorname{cotang.}\beta_1 = -\operatorname{cotang.}\alpha + \frac{\varepsilon^2}{\varepsilon - \lambda} \cdot \frac{1}{m \cdot \sin.2\alpha}.$$

We are now at the same point in the investigation as above, and we may therefore determine the remaining elements as indicated in the foregoing application. However, since the value of (m) has been assumed, the formula (7a) should be used to find the approximate value of the angle (β_2) which we have denoted by $[\beta_2]$.

The value of the absolute velocity (v) is also to be determined from Eq. (5) instead of Eq. (4) as was used above; and we thus have:

$$v = \sqrt{m \cdot 2gH}$$

The value of the velocity (v_1) in like manner also follows more directly from the formula:

$$v_1 = \frac{\varepsilon}{\cos.\alpha}\sqrt{\frac{2gH}{m}}.$$

From this point on, the remaining elements are found exactly as above.

In general it is to be recommended to design the Turbine as an Action wheel, so that the value of the characteristic (m) shall be:

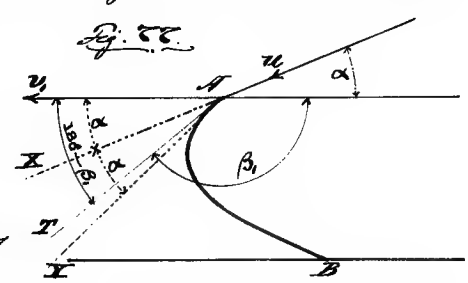
$$m = 1 - \varphi_0 = 0{,}8 \text{ to } 0{,}9.$$

It is also to be remarked here that in an Action Turbine, a certain dependence exists between the angles (α) and (β_1), which is, that:

$$180° - \beta_1 \geqq 2\alpha.$$

In such a Turbine, the value of the angle (α) must always be taken considerably less than $(45°)$, as will be seen from an examination of the formula (7a).

Fig. 77.



We shall accordingly have the following construction for the direction of the buckets in the moving wheel. Draw as in Fig. 77 the directions of the velocities (v) and (v_1) forming an angle (α). Produce the direction of (v) in AX, and then lay off AY so that the angle XAY shall be $=(\alpha)$, and hence the angle $v_1AY = 2\alpha$. Then draw AT so that v_1AT shall be less than 2α, and hence $180° - \beta_1 < 2\alpha$. The line AT will then be the tangent to the bucket curve AB at the point of entrance of the water.

In Action Turbines, we should also always make the expression:
$\frac{\varepsilon^2}{\varepsilon - \lambda}$ a little less than (m).

Generally, in the case of an action Full Turbine, we may assume:
$\eta = 0{,}7$; $\lambda = 0{,}05$, and: $\varepsilon = 0{,}75$;

hence we shall have: $\frac{\varepsilon^2}{\varepsilon - \lambda} = 0{,}8$, which is a little less than (m);
and therefore also:

$$\text{cotang.}\beta_1 \text{ somewhat less than: } < \text{cotang.}\alpha + \frac{1}{\sin.2\alpha},$$

or:

$$\text{cotang.}\beta_1 \quad -''- \quad -''- \quad -''-: < \frac{-2.\cos.^2\alpha + 1}{\sin.2\alpha},$$

or:

$$\text{cotang.}\beta_1 \quad -''- \quad -''- \quad -''- < \text{cotang.}2\alpha$$

or: $-\text{cotang.}\beta_1 \quad -''- > \text{cotang.}2\alpha$

But: $-\text{cotang.}\beta_1 = \text{cotang.}(180° - \beta_1) > \text{cotang.}2\alpha$,

or: $\text{tang.}(180° - \beta_1) < \text{tang.}2\alpha$,

whence: $180° - \beta_1 < 2\alpha$.

It may further be remarked that in such a Turbine, the ratio: $\left(\frac{w_1}{w_2}\right)$ of the relative entrance and discharge velocities (w_1) and (w_2) is very nearly equal to the ratio of the radii $\left(\frac{r_1}{r_2}\right)$ of the wheel at the points of entrance and discharge of the water. In short we may place:

$$\frac{w_1}{w_2} \text{ very nearly} = \frac{r_1}{r_2},$$

as will be seen in the following
From the general equation (1), page 91, we have:

$$w_1 = v_1 . \frac{\sin.\alpha}{\sin.(\alpha + \beta_1)};$$

and from eq. (2), same page:

$$w_2 = \frac{v_2}{\cos.\beta_2} = \frac{r_2}{r_1} \cdot \frac{v_1}{\cos.\beta_2}, \text{ since: } v_2 = \frac{r_2}{r_1} . v_1 .$$

Dividing the first of these equations by the second, we have:

$$\frac{w_1}{w_2} = \frac{r_1}{r_2} \cdot \frac{\sin.\alpha . \cos.\beta_2}{\sin.(\alpha + \beta_1)};$$

but in an action Turbine we have:

$180° - \beta_1$ only a little $< 2\alpha$,

whence: $180° - (\alpha + \beta_1)$ -''- $< \alpha$;

and because the expression $\left(\frac{\sin.\alpha}{\sin.(\alpha + \beta_1)}\right)$ is a proper fraction, we have:

$\frac{\sin.\alpha}{\sin.(\alpha + \beta_1)}$ only a little > 1

The angle (β_2) moreover is always very small, so that $(\cos.\beta_2)$ is only a little < 1; and therefore the product of two quantities, one of which is only a little greater than (1) and the other only a little less than (1), does not differ appreciably from (1) itself: and hence we may place:

$$\frac{\sin.\alpha . \cos.\beta_2}{\sin.(\alpha + \beta_1)} = 1 \text{ very nearly}$$

Accordingly in the case of an Action Turbine, we must have the condition that:

$$\frac{w_1}{w_2} \text{ shall be very nearly} = \frac{r_1}{r_2}.$$

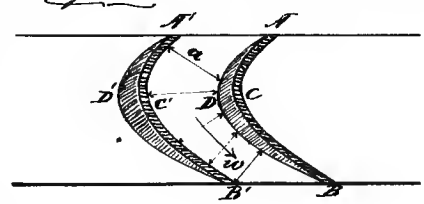
An Action Turbine has the advantage that the working coefficient (η) is not sensibly diminished when the wheel is only partially shut (Ex = -anfachlagt); whereas a Reaction Turbine suffers a very considerable loss in its useful effect when it is not fully shut. This loss is of course meant to be relatively to the amount of water (Q).

When an action Turbine is submerged, we have already seen that the thickness of the buckets in the moving wheel must be made variable in order that the canals shall be completely filled out with regularly flowing water, and hence that no empty spaces shall exist in these canals which would allow the water to set back into them, and thus by acting as a dead weight, cause a loss of effect to the wheel.

It therefore becomes necessary to know the law of this variation in thickness, (δ), and hence to find an analytical expression for this thickness.

For this purpose let us consider the case of a side shot Turbine, Fig. 78, in which ACB and $A'C'B'$ are any two consecutive buckets. At any point in this cross section, denote the dimension CD of the cross-section of the water volume by (a) and the other dimension by (b). The corresponding dimensions of the cross sections of the canals in the moving wheel at the points of entrance and discharge of the water were denoted by (a_1, b_1) and (a_2, b_2).

Fig. 78.



The dimension (b) may be considered as known, as also the law of the variation of this dimension from (b_1) to (b_2); and there thus remains to be calculated the dimension (a) for various points.

Denoting the relative velocity of the water in the canal by (w), we then have for the water volume which passes per second thro' the wheel:

$$a.b.w = a_1 b_1 . w_1 = a_2 b_2 . w_2 ,$$

assuming that the bucket space or canal shall be perfectly filled out with regularly flowing water (regelrecht strömendem wasser).

If we could assume, in the case of an action Turbine, that the law of variation for the dimension (a) was expressed by the relation:

$$\frac{w_1}{w_2} = \frac{r_1}{r_2} ,$$

we should then also have:

$$w : w_1 : w_2 = x : r_1 : r_2 .$$

This expression for the law of variation is however only approximate, and we will therefore assume that:

$$w = a + b.x ,$$

where (a) and (b) are constants.

On this assumption, we then also have:

$$w_1 = a + b.r_1 ,$$

and: $$w_2 = a + b.r_2 ,$$

from which there follows by the elimination of (a):

$$b = \frac{w_2 - w_1}{r_2 - r_1} ;$$

and from the value of (w_1): $a = w_1 - b.r_1$.

Substituting these values of (a) and (b) in the equation for (w), there follows:

$$w = w_1 - b r_1 + b x = w_1 + \frac{w_2 - w_1}{r_2 - r_1}(x - r_1);$$

and in like manner:

$$w = w_2 - b r_2 + b x = w_2 + \frac{w_2 - w_1}{r_2 - r_1}(x - r_2).$$

From these values of (w) the several values of the dimension (a) may then be calculated by means of Eq. (2) on the preceding page:

$$a b w = a_1 b_1 w_1 = a_2 b_2 w_2,$$

and the values thus found laid off at the respective points for the dimension (b). =

Connecting the terminal points of these values (a), we shall finally obtain the curves A D B, A' D' B', &c., which determine the thickness (b) of the buckets at various points; and when these curves and the canals in the runner wheel of each an action Turbine will be completed, filled and curved regularly for any given motion.

Different varieties of Full Turbines. — Let us now pass to the consideration of the various Kinds of Full Turbines, which may in general be divided into three classes: viz:— Inner-shot, — Outer-shot and Side-shot Full Turbines.

We will begin with the inner-shot wheels, which are the oldest, and are still known by the name of their inventor, or of the engineer who at all events first used these wheels with success, Fourneyron.

This Turbine was the result of a prize of 6000 francs offered by the French Scientific Society in the year 1826 to any one who should present a satisfactory design for such an hydraulic motor.

At the time set for the examination of the various designs in 1827, the plans of Burdin were alone deemed worthy of attention, and therefore to induce greater competition, the time was extended until 1829, when Fourneyron submitted his designs and obtained the prize.

In 1830, Fourneyron built and set up a turbine at the village of St. Blasien in the Black Forest. This machine, which existed until quite recently, performed with a head of 108 metres over 2000 revolutions per minute, and developed about 36 horse powers.

The first who established a satisfactory theory of Turbines was the French engineer Poncelet, who developed this theory with such accuracy, that all the present theories of Turbines, differing only in a few unimportant particulars, are based upon this.

In the year 1836, experiments upon Fourneyron's Turbine were made by Morin.

The practical execution (ausführung) of Fourneyron's Turbines has been somewhat modified of late years. Formerly it was always built with the moving apparatus deep in the lower water and enclosed in a tube so that access to the machine was extremely difficult. Now it is usually built in an inverted position to insure facility of inspection and repairs.

Fourneyrons Turbine is essentially a reaction wheel, the characteristic (nn) being about one-half: $n = 1/2$.

With the most favorable run, the working effect of this wheel has been found by experiment to be about:

$$\eta = 0,65.$$

For the required quantity of water we thus have:

$$Q = \frac{0,075}{\eta} \cdot \frac{N}{H} = 0,115 \cdot \frac{N}{H}.$$

From experiments it has also been found that under the most advantageous run of the turbine, the value of (λ) must be taken quite great, viz: $\lambda = 0,15$.

The hydraulic working coefficient (ε), in these turbines, must be taken from 0,06 to 0,08 greater than the value of (η), so that we shall have: $\varepsilon = 0,71$

In designing an inner-shot Fourneyron Full Turbine, it is usual to begin by assuming the values of the angles (α) and (β_1). In general we may place:

$$\alpha = 25° \text{ to } 30°.$$

This angle depends in a certain measure upon (Q) and (H), being taken greater the greater the value of (Q) and the less that of (H), as will be seen in the following.

The terminal cross-section of a canal in the guide wheel varies directly with the value of this angle (α); and hence the greater the value of (α), the greater also the total discharge cross-section (ausstritts querschnitt) of the entire guide wheel; and the smaller the value of (α), the smaller will this total area become. But this total cross-section or area must become greater, the greater the quantity of water (Q) which shall flow through this area, and the less the pressure or head (H) of the water; and we therefore see that the above principle is true.

Relatively to the angle (β_1), its value was always assumed by Fourneyron to be: $\beta_1 = 90°$.

Redtenbacher, on the other hand, has recommended that its value should not be taken greater than:

$$\beta_1 = 60°.$$

On this latter assumption, the curvature of the buckets will be much less than on the former, as is seen from Fig. 79; but this value of (β_1) has another disadvantage.

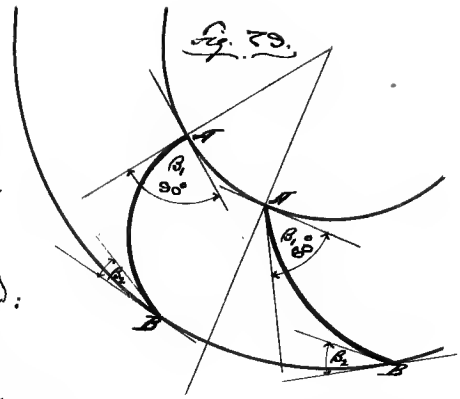


Fig. 79.

For if we substitute the above values in the equation (b) on page 99:

$$\frac{\varepsilon-\lambda}{\varepsilon^2} m. \sin.2\alpha \left(\text{Cotg.}\,\alpha + \text{cotg.}\,\beta_1\right) = 1$$

and deduce from this the value of the characteristic (m), we shall obtain when the value given by Fourneyron to (β_1) is used:

$$m = 0,548 ;$$

and when Redtenbacher's value for (β_1) is used:

$$m = 0,431 .$$

In this latter case the value of (m) is thus much less than ½, and the turbine constructed with this value of (β_1) will accordingly approach nearer to a reaction than to an action wheel as in the former case, and will therefore also become more dependent relatively to its useful effect upon the head (Gefälle) and quantity of the water supply.

It therefore becomes advisable to retain the first value of the angle (β_1), which was assigned to it by Fourneyron, viz: $\beta_1 = 90°$, rather than to diminish the characteristic so much.

From the above values of (ε) and (η), the value of the coefficient (φ) may now also be found. Taking the coefficient (μ) of the axles friction work $= 0,04$, which is determined experimentally, and (ε) about 0,06 greater than (η), we then have from the formulas:

$$\varepsilon = \frac{\eta + \mu}{\varphi} ,$$

$$\varphi = 0,98 .$$

With this value of (φ), the coefficient (K) is determined from: